# Math 4121 Midterm 2

March 27, 2019

Name: ______________________________

- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
  - Make a note on the printed page that your work continues on the back of the previous page.
  - On the back of the previous page, put a box around the work that you want graded.
- Give and use definitions from the book or from class.
- You may use any results you remember from the book or from class as long as they are more basic than the result you're asked to prove.

1. (a) (5 points) State the Monotone Convergence Theorem.

   (b) (5 points) State the Dominated Convergence Theorem.

   (c) (5 points) State Fatou's Lemma (either version in the textbook).

2. (20 points) Give an example of functions $f_n$ and $f$ defined on $[a, b]$ such that

   - $f_n$ and $f$ are in $L$ and
   - $f_n \to f$ a.e., but
   - $\int f_n \not\to \int f$.

   Make sure to at least briefly explain why your example satisfies each condition.

3. Let $f_n$ be a sequence of measurable functions on $[a, b]$. Assume that for almost every $x$ we have that $\sup\{f_n(x) \mid n \in \mathbb{N}\}$ exists, and define $f = \sup_n f_n$.

   (a) (10 points) Show that $f$ is measurable.

   (b) (10 points) Assume furthermore that there exists an integrable function $g$ such that for all $n$ we have that $|f_n| \leq g$ almost everywhere. Show that $f$ is integrable.

   (c) (10 points) Give an example of functions $f_n$ such that each $f_n$ is integrable and $f = \sup_n f_n$ exists almost everywhere, but $f$ is not integrable.

   Make sure to at least briefly explain why your example satisfies each condition.

4. In this question, you will prove the following proposition in two different ways.

   **Proposition** (Continuity of measure from below)**.** *Let $E_n$ be a sequence of measurable subsets of $[a, b]$, nested so that*

$$E_1 \subseteq E_2 \subseteq \cdots .$$

   *Let $E = \bigcup_{n=1}^{\infty} E_n$. Then*

$$m(E) = \lim_{n\to\infty} m(E_n).$$

   (a) i. (5 points) State without proof the property of measure called *countable additivity.*

   ii. (15 points) Prove the above claim using countable additivity.

   (b) (15 points) Prove the above claim using the Monotone Convergence Theorem.