Math 4121 Midterm 1

February 13, 2019

Name: ______________________________________

- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
  - Make a note on the printed page that your work continues on the back of the previous page.
  - On the back of the previous page, put a box around the work that you want graded.
- Give and use definitions from the book or from class.
- You may use any results you remember from the book or from class as long as they are more basic than the result you're asked to prove.

1. Let $f\colon [a,b] \to \mathbb{R}$.

(a) i. (5 points) Define what it means to be a Riemann sum for f (also referred to by the textbook as a Cauchy sum).

ii. (5 points) Define what it means for f to be Riemann integrable.

Solution: Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$. Let $\xi_k \in [x_{k-1}, x_k]$. A Riemann sum for f is a sum of the form

$$S(P; f) = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}).$$

The function f is Riemann integrable if

$$\lim_{|P|\to 0} S(P; f)$$

exists. That is, there exists an L such that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|S(P; f) - L| < \epsilon$$

for any partition P with $|P| < \delta$ and any choice of sample points ξ_k.

(b) i. (4 points) Define what it means to be an upper Darboux sum for f.

ii. (4 points) Define the upper Darboux integral.

iii. (4 points) State Darboux's criterion for when f is Riemann integrable.

Solution: With notation as above, let

$$M_k = \sup_{x\in[x_{k-1},x_k]} f(x).$$

The upper Darboux sum for f with respect to the partition P is

$$\bar{S}(P; f) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}).$$

The upper Darboux integral is

$$\overline{\int_a^b} f(x)\,dx = \inf\{\bar{S}(P; f) \mid P \text{ is a partition of } [a,b]\}.$$

Darboux's criterion for Riemann integrability states that f is Riemann integrable if and only if

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx.$$

(c) i. (4 points) Define measure zero.

ii. (4 points) State Lebesgue's criterion for when f is Riemann integrable.

Solution: A subset $A \subseteq \mathbb{R}$ has measure zero if for all $\epsilon > 0$, there exists a countable collection of intervals $\{I_n\}_{n=1}^{\infty}$, such that

$$A \subseteq \bigcup_{n=1}^{\infty} I_n,$$

and

$$\sum_{n=1}^{\infty} |I_n| < \epsilon.$$

Lebesgue's criterion for Riemann integrability states that f is Riemann integrable if and only if f is bounded and the set of discontiuities of f has measure zero.

2. Let $f\colon [-1,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

(a) (15 points) Show that f is Riemann integrable using either the definition or Darboux's criterion, but without using Lebesgue's criterion.

Solution: For each $\epsilon > 0$, define a partition $P = \{-1 < -\epsilon < \epsilon < 1\}$. Then

$$\overline{S}(P; f) = (-1)(1-\epsilon) + (1)(2\epsilon) + (1)(1-\epsilon) = 2\epsilon,$$
$$\underline{S}(P; f) = (-1)(1-\epsilon) + (-1)(2\epsilon) + (1)(1-\epsilon) = -2\epsilon.$$

Since $\underline{S}(P; f) \le \underline{\int}_{-1}^{1} f(x)\,dx$ and $\overline{\int}_{-1}^{1} f(x)\,dx \le \overline{S}(P; f)$, we conclude that

$$-2\epsilon \le \underline{\int}_{-1}^{1} f(x)\,dx \le \overline{\int}_{-1}^{1} f(x)\,dx \le 2\epsilon.$$

Since ϵ was an arbitrary positive number, we conclude that

$$0 = \underline{\int}_{-1}^{1} f(x)\,dx = \overline{\int}_{-1}^{1} f(x)\,dx = 0.$$

By Darboux's criterion, since the upper and lower Darboux integrals agree, f is Riemann integrable.

(b) (5 points) Show that f is Riemann integrable using Lebesgue's criterion.

Solution: f is continuous except at $x = 0$. The set of discontinuities of f is a single point $\{0\}$, which is a measure zero set. Thus, f is Riemann integrable.

3. (10 points) Show that a countable union of measure zero sets has measure zero.

Solution: Let $A = \bigcup_{j=1}^{\infty} A_j$, where each A_j has measure zero. For all $\epsilon > 0$, let $\epsilon_j = \epsilon \cdot 2^{-j}$. Since A_j has measure zero, there exist intervals $\{I_{j,n}\}_{n=1}^{\infty}$ such that

$$A_j \subseteq \bigcup_{n=1}^{\infty} I_{j,n},$$

and

$$\sum_{n=1}^{\infty} |I_{j,n}| < \epsilon_j.$$

But then $\{I_{j,n} \mid j, n \in \mathbb{N}\}$ is a countable collection of intervals, and

$$A = \bigcup_{j=1}^{\infty} A_j \subseteq \bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} I_{j,n},$$

and

$$\sum_{j,n} |I_{j,n}| = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} |I_{j,n}| < \sum_{j=1}^{\infty} \epsilon_j = \epsilon.$$

Thus, A has measure zero by definition.

4. Let $f\colon [a,b] \to \mathbb{R}$.

(a) (5 points) Let $c \in [a,b]$. Define the limit superior of f at c.

Solution: The limit superior of f at c is defined by

$$\begin{aligned}\limsup_{x\to c} f(x) &= \lim_{\delta\to 0}\big(\sup\{f(x) \mid x \in (c-\delta, c+\delta) \text{ and } x \in [a,b]\}\big)\\ &= \inf_{\delta>0}\big(\sup\{f(x) \mid x \in (c-\delta, c+\delta) \text{ and } x \in [a,b]\}\big).\end{aligned}$$

(b) (5 points) Define the oscillation of f at c.

Solution: The oscillation of f at c is defined by

$$\omega(f;c) = \limsup_{x\to c} f(x) - \liminf_{x\to c} f(x).$$

(c) (20 points) Show that f is continuous at c if and only if the oscillation of f at c is zero.

Solution: Assume that f is continuous at c. Then, for all $\epsilon > 0$, there exists a $\delta > 0$ such that $f(c) - \epsilon < f(x) < f(c) + \epsilon$ for all $x \in (c-\delta, c+\delta) \cap [a,b]$. Thus,

$$\begin{aligned}f(c) - \epsilon \le \inf\{f(x) \mid x \in (c-\delta, c+\delta) \cap [a,b]\} &\le \liminf_{x\to c} f(x)\\ \le \limsup_{x\to c} f(x) \le \sup\{f(x) \mid x \in (c-\delta, c+\delta) \cap [a,b]\} &\le f(c) + \epsilon.\end{aligned}$$

Since this inequality is true for all ϵ and $\liminf_{x\to c} f(x)$ and $\limsup_{x\to c} f(x)$ does not depend on ϵ, we conclude that

$$f(c) \le \liminf_{x\to c} f(x) \le \limsup_{x\to c} f(x) \le f(c).$$

We conclude that $\liminf_{x\to c} f(x) = \limsup_{x\to c} f(x)$, so $\omega(f;c) = 0$.
Conversely, assume that $\omega(f;c) = 0$, so $\liminf_{x\to c} f(x) = f(c) = \limsup_{x\to c} f(x)$. Then for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\begin{aligned}f(c) - \epsilon &< \inf\{f(x) \mid x \in (c-\delta, c+\delta) \cap [a,b]\}\\ &\le \sup\{f(x) \mid x \in (c-\delta, c+\delta) \cap [a,b]\} < f(c) + \epsilon.\end{aligned}$$

In other words, for all $x \in (c-\delta, c+\delta) \cap [a,b]$, we have $f(c) - \epsilon < f(x) < f(c) + \epsilon$. We conclude that f is continuous at c.

5. (10 points) Give an example of a bounded function f that is not Riemann integrable. Justify that your example is not Riemann integrable using any results from class.

Solution: Let $f\colon [0,1] \to \mathbb{R}$ be the Dirichlet function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is discontinous everywhere. Indeed, for any x and any $\delta > 0$, there exists both a rational and an irrational point in $(x - \delta, x + \delta)$, so we can choose $y \in (x - \delta, x + \delta)$ such that $|f(y) - f(x)| = 1$.

The set of discontinuities of f is $[0, 1]$. We showed in class that the closed interval $[a, b]$ with $a < b$ does not have measure zero. Lebesgue's criterion then says that because the set of discontinuities of f does not have measure zero, f cannot be Riemann integrable.