Math 4121 Midterm 1

February 13, 2019

Name: ____

- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
  - Make a note on the printed page that your work continues on the back of the previous page.
  - On the back of the previous page, put a box around the work that you want graded.
- Give and use definitions from the book or from class.
- You may use any results you remember from the book or from class as long as they are more basic than the result you're asked to prove.

1. Let $f\colon [a,b] \to \mathbb{R}$.
   (a) i. (5 points) Define what it means to be a Riemann sum for f (also referred to by the textbook as a Cauchy sum).
   ii. (5 points) Define what it means for f to be Riemann integrable.

   (b) i. (4 points) Define what it means to be an upper Darboux sum for f.
   ii. (4 points) Define the upper Darboux integral.
   iii. (4 points) State Darboux's criterion for when f is Riemann integrable.

   (c) i. (4 points) Define measure zero.
   ii. (4 points) State Lebesgue's criterion for when f is Riemann integrable.

2. Let $f\colon [-1,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

(a) (15 points) Show that f is Riemann integrable using either the definition or Darboux's criterion, but without using Lebesgue's criterion.

(b) (5 points) Show that f is Riemann integrable using Lebesgue's criterion.

3. (10 points) Show that a countable union of measure zero sets has measure zero.

4. Let $f\colon [a,b] \to \mathbb{R}$.

   (a) (5 points) Let $c \in [a,b]$. Define the limit superior of f at c.

   (b) (5 points) Define the oscillation of f at c.

   (c) (20 points) Show that f is continuous at c if and only if the oscillation of f at c is zero.

5. (10 points) Give an example of a bounded function f that is not Riemann integrable. Justify that your example is not Riemann integrable using any results from class.