HOMEWORK 9

MATH 4121

(1) (Text, Exercise III.7.A, page 111.) Show that a sequence (f_n) of functions on $[0,1]$ defined by $f_n(x) = x^n$ converges almost uniformly without using the Egoroff's theorem.

(2) Draw the diagram of page 110 summarizing the relations between the different kinds of convergences of sequences of measurable functions on $[a,b]$ and add to it arrows drawn in dashed lines

$$\text{convergence mode } A \dashrightarrow \text{convergence mode } B$$

when convergence of a sequence in A implies convergence of a subsequence in B.

(3) Let $f_n \in L^1[a,b]$. Show that if $f_n \to f$ uniformly, then $f_n \to f$ in L^1.

(4) Give and justify an example of bounded functions f_n and f in L^1 such that $f_n \to f$ in L^1, but $f_n \not\to f$ uniformly.

(5) Give and justify an example of functions f_n and f in L^1 such that $f_n \to f$ in measure but $f_n \not\to f$ in L^1.

(6) (Text, Exercise IV.1.E, page 132.) Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} (-1)^{n+1} n & \text{if } \frac{1}{n+1} < x \leq \frac{1}{n}, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is not Lebesgue integrable on $[0,1]$, but it is improperly Riemann integrable. The improper integral is equal to $1 - \log 2$.