HOMEWORK 8

MATH 4121

(1) (Text, Exercise III.3.D, page 97.) Let $f : [a, b] \to \mathbb{R}^*$ be measurable. Show that if $\mathcal{C}$ is a collection of subsets E of $\mathbb{R}^*$ for which $f^{-1}(E)$ is measurable, then $\mathcal{C}$ is a σ-algebra.

(2) (Text, Exercise III.3.E, page 97.) Let $f : [a, b] \to \mathbb{R}^*$ be measurable and let B be a Borel set. Show that $f^{-1}(B)$ is measurable. Hint: Exercise D.

(3) (Text, Exercise III.4.E and III.4.C, page 100). A function $\varphi : \mathbb{R} \to \mathbb{R}$ is called an *isometry* if

$$|\varphi(x) - \varphi(y)| = |x - y|$$

for all x, y.

(a) Show that every isometry has one of the following forms:

$$\varphi(x) = x + c \quad \text{(translation)},$$
$$\varphi(x) = -x \quad \text{(reflection in the origin)},$$
$$\varphi(x) = -x + c.$$

(b) Show that if A is a measurable set in $[a, b]$ and if φ is an isometry, then $\varphi(A)$ is measurable in the interval whose endpoints are $\varphi(a)$ and $\varphi(b)$, and

$$m(\varphi(A)) = m(A).$$

(c) Let $A \subset [a, b]$ be measurable. For any number $r \neq 0$, let $rA = \{rx : x \in A\}$. Show that rA is measurable and

$$m(rA) = |r|m(A).$$

Feel free to use results from III.5 in your solution even though this is a problem from III.4.

(4) (Text, Exercise III.5.C, page 103.) Show that a set A is measurable if and only if, for every $\epsilon > 0$, there exists both an open set G and a closed set F such that $F \subset A \subset G$ and

$$m(G \setminus F) < \epsilon.$$

(5) (Text, Exercise III.6.A, page 105.) Show that a function f on $[a, b]$ is measurable if and only if the set $\{x \in [a, b] : f(x) > r\}$ is measurable for each rational number r. (In fact, for all r in any set $Q \subset \mathbb{R}$ which is dense in $\mathbb{R}$.)

(6) (Text, Exercise III.6.B and C, pages 105, 106.) Let f be a measurable function.

(a) If g is also a measurable function, show that $\{x \in [a,b] : f(x) \leq g(x)\}$ is a measurable set.

(b) If $g = f$ almost everywhere, then g is a measurable function.

(7) (Text, Exercise III.6.D, page 106.) If f is such that $|f|$ is measurable, does f have to be measurable?