HOMEWORK 6

MATH 4121

(1) (Exercise II.6.A, page 78 of textbook.) Show that there is no sequence of functions on $[0, 2\pi]$ of the type

$$f_n(x) = a_n \sin nx + b_n \cos nx$$

which converges to the function 1 almost everywhere on $[0, 2\pi]$, and where $|a_n| + |b_n| \leq 10$.

(2) Show that

$$\int_0^1 \left(\frac{\log x}{1-x}\right)^2 dx = \frac{\pi^2}{3}.$$

I suggest the following approach:

(a) First show that for all $x \in (-1, 1)$

$$\sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}.$$

(b) Define $f_n(x) = n x^{n-1} (\log x)^2$ for $n \geq 1$ and $x \in (0, 1)$. Show that $f_n(x)$ belongs to the class L.

(c) Now show that

$$\int_0^1 x^{n-1} (\log x)^2 \, dx = \frac{2}{n^3}.$$

(Integration by parts!)

(d) Justify that we are allowed to calculate the integral of $f(x) = \left(\frac{\log x}{1-x}\right)^2$ over $[0, 1]$ by integrating the series term by term:

$$\int_0^1 f(x) \, dx = \sum_{n=1}^{\infty} \int_0^1 f_n(x) \, dx.$$

(e) Show that the integral of $f(x)$ is $\pi^2/3$. You may take for granted that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

You need not prove this well-known fact. See `https://en.wikipedia.org/wiki/Basel_problem`.

(3) (Exercise II.6.D, page 78 of textbook.) Let (f_n) be a sequence of functions in the class L and let $f \in L$ be such that

$$\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|\, dx = 0.$$

Show that if $f_n \to g$ almost everywhere, then $f = g$ almost everywhere.

(4) (Exercise II.6.H, page 79 of textbook.) Prove that if (f_n) is a sequence in L which converges almost everywhere to a function f, and $|f| \leq g$ for some $g \in L$, then $f \in L$.

(5) (Exercise II.7.D, page 84 of textbook.) Let f be integrable on $[a, b]$, and let $\epsilon > 0$. Show that there is an indefinitely differentiable function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\int_a^b |f(x) - g(x)|\, dx < \epsilon.$$

Hint: One approach uses the Stone-Weierstrass theorem.

(6) (Text, Exercise III.1.A, page 90.) Show that the function f defined by $f(x) = 1/x$ if $x \in (0, 1]$ and $f(0) = 0$ is not integrable on $[0, 1]$.