HOMEWORK 2

MATH 4121

(1) Consider these three sequences of functions:

$$A_n(x) = nxe^{-nx^2}, \quad B_n(x) = \frac{n^2 x}{1+n^3x^2}, \quad C_n(x) = \frac{nx}{1+n^2x^2}$$

for $n \in \mathbb{N}$. Show that the following statements hold by computing the various limits and integrals.

(a) $\lim_{n\to\infty} \int_0^1 A_n(x)\,dx \neq \int_0^1 \lim_{n\to\infty} A_n(x)\,dx$

(b) $\lim_{n\to\infty} \int_0^1 B_n(x)\,dx = \int_0^1 \lim_{n\to\infty} B_n(x)\,dx$

(c) $\lim_{n\to\infty} \int_0^1 C_n(x)\,dx = \int_0^1 \lim_{n\to\infty} C_n(x)\,dx$

(d) Explain: A_n, B_n and C_n converge pointwise to (the function identically equal to) 0, but not uniformly.

(e) Which of the three families is (or are) uniformly bounded? Justify.

(2) **Baire's example.** Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} 1 & \text{if } x = 0 \text{ or } x = \frac{p}{q}, \text{ with } p, q \in \mathbb{N} \text{ and } q \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Explain:

(a) $f_n(x)$ is integrable on $[0,1]$ for all n and $\lim_{n\to\infty} \int_0^1 f(x)\,dx = 0$.

(b) $f_n(x)$ converges pointwise to a bounded function $f(x)$ on $[0,1]$ which is not integrable.