HOMEWORK 11

MATH 4121

(1) (Text, Exercise IV.5.C, page 151.) Let f be integrable on $[0,1]\times[0,1]$. Show that

$$\int_0^1 \left[\int_0^x f(x,y)\,dy\right] dx = \int_0^1 \left[\int_y^1 f(x,y)\,dx\right] dy.$$

(2) (Text, Exercise IV.5.H, page 151.) Compute the following integrals. *Hint:* See the above exercise.

(a) $\int_0^1 \left[\int_y^1 \exp\left(-x^2\right)\,dx\right] dy$;

(b) $\int_0^1 \left[\int_y^1 ((\sin x)/x)\,dx\right] dy$; and

(c) $\int_0^1 \left[\int_y^1 \sin x^2\,dx\right] dy$.

(3) (Text, Exercise IV.5.D, page 151.) Let

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^2} & \text{if } (x,y)\neq(0,0), \\ 0 & \text{if } (x,y)=(0,0). \end{cases}$$

(a) Show that

$$\int_0^1 \left[\int_0^1 f(x,y)\,dy\right] dx = \frac{\pi}{4},$$
$$\int_0^1 \left[\int_0^1 f(x,y)\,dx\right] dy = -\frac{\pi}{4}.$$

(b) Conclude from the Fubini theorem that f is not integrable on $[0,1]\times[0,1]$.

(4) (Text, Exercise IV.5.F, page 151.) Let $R=[-1,1]\times[-1,1]$ and let

$$f(x,y) = \begin{cases} \frac{xy}{(1-|x|)^2+(1-|y|)^2} & \text{if } |xy|\neq 1, \\ 0 & \text{if } |xy|=1. \end{cases}$$

(a) Show that

$$\int_{-1}^1 \left[\int_{-1}^1 f(x,y)\,dx\right] dy = \int_{-1}^1 \left[\int_{-1}^1 f(x,y)\,dy\right] dx = 0.$$

(b) Show that f is not integrable on R. *Hint:* f is not integrable on $\left[\frac{1}{2},1\right]\times\left[\frac{1}{2},1\right]$.

(5) (Text, Exercise IV.5.G, page 151.) Let f and g be measurable functions on $[a,b]$. Show that the function $(x,y)\mapsto f(x)g(y)$ is measurable on $[a,b]\times[a,b]$.

(6) (Text, Exercise V.1.D, page 159.) *Van der Waerden's Example of a Continuous Nowhere Differentiable Function* (Van der Waerden, 1930). Let

$$\varphi_0(x) = \begin{cases} x & \text{if } 0 \le x \le \frac{1}{2} \\ 1-x & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Extend φ_0 by periodicity with period 1 to the whole line $\mathbb{R}$. Then let

$$\varphi_n(x) = \frac{\varphi_0\left(4^n x\right)}{4^n}, \quad n = 1, 2, \ldots, \text{ and } \quad f(x) = \sum_{n=0}^{\infty} \varphi_n(x).$$

Show that:
(a) f is continuous everywhere; and
(b) f is nowhere differentiable.
Hint: Consider

$$\frac{f\left(x_0 \pm 4^{-n}\right) - f(x_0)}{4^{-n}}.$$

(7) (Text, Exercise V.4.D, page 177.) Suppose that f' exists everywhere and is bounded on $[a, b]$. Show that f is of bounded variation. *Hint:* The mean value theorem.

(8) (Text, Exercise V.5.B, page 183.) Show that the Lebesgue singular function described in §4 (Chapter V, Example 4.9, page 175) is not absolutely continuous. *Hint:* The Cantor ternary set F is compact and is of measure zero.

(9) (Text, Exercise V.6.A, page 189.) Let

$$f(x) = \begin{cases} x^2 \sin \pi/x^2 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

Show that f' exists on $(0, 1)$, but f' is not integrable on $[0, 1]$. *Hint:*

$$\int_0^1 \frac{1}{x} \left|\cos \frac{\pi}{x^2}\right| \, dx = \infty.$$

(10) (Text, Exercise V.6.C, page 189, 10 points.) Let f be integrable on $[a, b]$ and

$$\int_a^x f(t)\, dt = 0$$

for all $x \in [a, b]$. Show that $f = 0$ almost everywhere.