HOMEWORK 10

MATH 4121

(1) (Text, Exercise IV.1.F, page 132.) Let f be as in IV.1.E. Show that for any real number r there is a sequence (E_n) of mutually disjoint measurable sets (namely, intervals) in $[0,1]$ such that $[0,1] = \bigcup_{n=1}^{\infty} E_n$ and

$$\sum_{n=1}^{\infty} \int_{E_n} f(x)\,dx = r.$$

(2) (Text, Exercise IV.2.B, page 135.) Give an example of a nonintegrable function f whose absolute value $|f|$ is integrable on $\mathbb{R}$. (This shows that the converse of Corollary 2.5, Chapter IV, may fail.)

(3) (Text, Exercise IV.2.C, page 135.) Which of the following functions are integrable on $[0,\infty)$?
   (a) The characteristic function of the rationals in $[0,\infty)$.
   (b) The characteristic function of the irrationals in $[0,\infty)$.

(4) (Text, Exercise IV.2.D, page 135.) If f is nonnegative and improper Riemann integrable on $\mathbb{R}$, prove that f is Lebesgue integrable.

(5) (Text, Exercise IV.2.E, page 135.) Show that if $x > 0$, then the function

$$t \mapsto e^{-t} t^{x-1}$$

is integrable on $[0,\infty)$. Further, show that

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{x-1}\,dt = \int_0^\infty e^{-t} t^{x-1}\,dt.$$

This function is known as the *gamma function*.

(6) Give a counterexample to Proposition IV.3.6 on page 138 in the case where $m(E_n) = \infty$ for all n. Specifically, construct subsets $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$ of $\mathbb{R}$ such that $m(E_n) = \infty$ for all n but $m(E) < \infty$, where $E = \bigcap_{n=1}^{\infty} E_n$.