HOMEWORK 1

MATH 4121

(1) **Textbook, I.3.B, page 34**. Show that if $f : [a, b] \to \mathbb{R}$ is a Riemann integrable function defined on the interval $[a, b]$, then f is bounded.

(2) **Textbook, I.3.C, page 34**. Show that the Dirichlet function is not Riemann integrable. Recall that the Dirichlet function $D : [0, 1] \to \mathbb{R}$ is defined as

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(3) **Textbook, I.3.D, page 34**. Prove that every continuous function on $[a, b]$ is Riemann integrable on $[a, b]$.

(4) **Textbook, I.3.H, page 35**. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ where } p, q \text{ are coprime positive integers and } q \neq 0 \\ 0 & \text{if } x \notin \mathbb{Q} \text{ or } x = 0, 1. \end{cases}$$

(a) Prove that f is continuous at every irrational point of $[0, 1]$ and discontinuous where $f(x) \neq 0$.

(b) Prove that f is Riemann integrable on $[0, 1]$ and $\int_0^1 f(x)\, dx = 0$.

(5) **Riemann's example.** To demonstrate the value of his formulation of the integral, Riemann (1854) proposed the following example of a function which is discontinuous on a dense set of points but still integrable. (The previous problem gives a simpler but less interesting example of such a function.)

First define $\langle x \rangle$ = the integer closest to x (and for definiteness let us require it to be right-continuous, so that $\langle 1/2 \rangle = 1$ for example. It is not important how it is defined on the points of discontinuity.) Now set

$$B(x) = \begin{cases} x - \langle x \rangle & \text{if } x \neq k/2 \text{ for any } k \in \mathbb{Z} \\ 0 & \text{if } x = k/2 \text{ for some } k \in \mathbb{Z} \end{cases}$$

and finally define Riemann's function as

$$f(x) = \sum_{n=1}^{\infty} \frac{B(nx)}{n^2}.$$

The graph of this function over $[0, 1]$ is shown in figure 1.

(a) Show that f is discontinuous at every rational point with an even denominator (and odd numerator).

(b) Show that f is integrable over $[0,1]$.

(c) Show that

$$\int_0^{1/2} f(x)\,dx = \frac{1}{8}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3} \approx 0.131475.$$

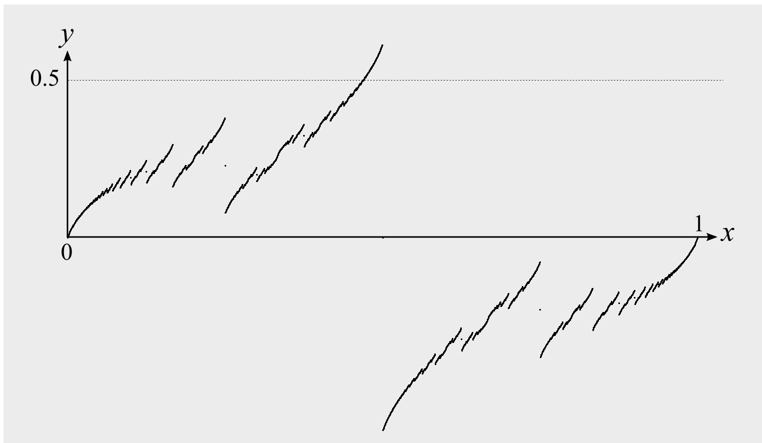

FIGURE 1. Graph of Riemann's function.

Remark. The following observations may be useful regarding Riemann's example (and others!). They are easily proved. First, it is an easy fact to prove fact that if $f(x)$ is Riemann integrable on $[a,b]$ then it is Riemann integrable on any subinterval of $[a,b]$. Second, let $a<b<c$ and suppose that $f:[a,c]\to\mathbb{R}$ is a function whose restrictions to $[a,b]$ and $[b,c]$ are integrable. Then f is integrable over $[a,c]$ and

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Conversely, integrability over $[a,c]$ implies integrability over subintervals. This and other elementary properties of the Riemann integral will be discussed in class.

The following theorem gives a sufficient condition for passing a limit across the integral sign. (Much more general results of this kind will be seen for the Lebesgue integral.)

Theorem. *Consider a sequence $f_n(x)$ of integrable functions and suppose that it converges uniformly on $[a,b]$ to a function $f(x)$. Then $f:[a,b]\to\mathbb{R}$ is integrable and*

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proof. Let us first check that f is integrable. Uniform convergence of the sequence implies, for any given $\epsilon > 0$, the existence of an integer N such that for all $n \geq N$ all all $x \in [a, b]$ one has $|f_n(x) - f(x)| < \epsilon$. Thus for all $x, y \in [a, b]$, an application of the triangle inequality gives

$$|f(x) - f(y)| \leq |f_N(x) - f_N(y)| + 2\epsilon. \tag{1}$$

Since f_N is integrable, there exists a partition of $[a, b]$ such that the difference between its upper and lower Darboux sums is less than ϵ. This means that for this partition, denoted $(x_0, \ldots, x_n)$, we have the following inequalities, where we write $\overline{f_i} = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$, $\underline{f_i} = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$:

$$\left(\overline{f_i} - \underline{f_i}\right) \leq \left(\overline{f_{Ni}} - \underline{f_{Ni}}\right) + 2\epsilon,$$

which follows from inequality (1) and, by the integrability of f_N,

$$\sum_i \left(\overline{f_{Ni}} - \underline{f_{Ni}}\right)(x_i - x_{i-1}) < \epsilon.$$

Consequently,

$$\sum_i \left(\overline{f_i} - \underline{f_i}\right)(x_i - x_{i-1}) < (1 + 2(b-a))\epsilon$$

and $f(x)$ is integrable. With the integrability of $f(x)$ assured, we can conclude that for all $n \geq N$,

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| \leq \int_a^b |f_n(x) - f(x)|\,dx \leq (b-a)\epsilon.$$

This concludes the proof. □