Math 4121 Final

May 6, 2019

Name: ____

- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
  - Make a note on the printed page that your work continues on the back of the previous page.
  - On the back of the previous page, put a box around the work that you want graded.
- Give and use definitions from the book or from class.
- You may use any results you remember from the book or from class as long as they are more basic than the result you're asked to prove.

1. (a) (3 points) State Lebesgue's criterion for when a function is Riemann integrable.

Solution: A function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable if and only if it is bounded and continuous almost everywhere.

(b) (3 points) Give and briefly justify an example of a function that is Lebesgue integrable but not Riemann integrable.

Solution: The Dirichlet function $D = \chi_{\mathbb{Q}\cap[0,1]}$ is Lebesgue integrable on $[0,1]$ because $D = 0$ almost everywhere. On the other hand, D is not continuous at any point x. Indeed, for any $x \in [0,1]$, set $\epsilon = 1$. Any δ-neighborhood of x contains both rational and irrational points, so $D(x-\delta, x+\delta)$ contains both 0 and 1 and is thus not contained in an ϵ-neighborhood of $D(x)$. We conclude that D is continuous nowhere, so it is not Riemann integrable.
Alternatively, let $f(x) = \frac{1}{\sqrt{x}}$ for $x \in (0,1]$ and $f(0) = 0$. This function is unbounded, which we showed in the homework implies that f is not Riemann integrable. On the other hand, we show later in the exam that f is Lebesgue integrable.

2. (a) (3 points) State the Monotone Convergence Theorem.

Solution: If $f_1 \leq f_2 \leq \cdots$ is a monotone sequence of Lebesgue integrable functions and $\int f_n$ is bounded above, then f_n converges almost everywhere to a Lebesgue integrable function f, and $\int f_n$ converges to $\int f$.

(b) (5 points) Show that $f\colon [0,1] \to \mathbb{R}^*$ defined by $f(x) = \frac{1}{\sqrt{x}}$ is integrable.

Solution: Let $f_n = f\chi_{[\frac{1}{n},1]}$. Then $f_n(x) \to f(x)$ except at $x = 0$. Also, f_n is bounded and continuous except at $\frac{1}{n}$, so it is Riemann integrable and hence Lebesgue integrable. We can compute that

$$\int f_n = \int_{\frac{1}{n}}^{1} \frac{1}{\sqrt{x}} = 2\sqrt{x}\Big|_{\frac{1}{n}}^{1} = 2 - \frac{2}{\sqrt{n}}.$$

Thus $\int f_n$ is bounded above by 2, and so we can apply the Monotone Convergence Theorem to conclude that the limit f is Lebesgue integrable.

(c) (5 points) Show that $f\colon [0,1] \to \mathbb{R}^*$ defined by $f(x) = \frac{1}{x}$ is not integrable.

Solution: Let $f_n = f\chi_{[\frac{1}{n},1]}$. Then $f_n \leq f$. Also, f_n is bounded and continuous except at $\frac{1}{n}$, so it is Riemann integrable and hence Lebesgue integrable. We can compute that

$$\int f_n = \int_{\frac{1}{n}}^{1} \frac{1}{x} = \ln x\Big|_{\frac{1}{n}}^{1} = -\ln \tfrac{1}{n} = \ln n.$$

Assume for the sake of contradiction that f is Lebesgue integrable. Then $f_n \leq f$ implies that $\int f_n \leq \int f$. But $\int f_n = \ln n$ is unbounded as $n \to \infty$, so we have a contradiction.

3. (a) (3 points) Define what it means for a function $f\colon [a,b] \to \mathbb{R}^*$ to be measurable.

Solution: A function f is measurable if there exists a sequence ϕ_n of step functions such that $\phi_n \to f$ almost everywhere.

(b) (3 points) Show that if f is measurable then $|f|$ is measurable.

Solution: If ϕ_n is a sequence of step functions that converges to f almost everywhere, then $|\phi_n|$ is a step function, and $|\phi_n| \to |f|$ almost everywhere. Thus $|f|$ is measurable by definition.

(c) (3 points) Give and justify an example of a function f such that $|f|$ is measurable but f is not measurable. You may assume the existence of a nonmeasurable subset $A \subseteq [a,b]$.

Solution: Let $f = \chi_A - \chi_{A^c}$. Then $|f| = 1$, which is measurable. Because A is not measurable, we know that χ_A is not measurable by definition. Observe that $1 = \chi_A + \chi_{A^c}$, so $f + 1 = 2\chi_A$. If f were measurable, then by properties of measurable functions, $f + 1$ would be measurable as well, as would $\frac{f+1}{2} = \chi_A$, a contradiction. Alternatively, if f were measurable, then by properties of measurable functions, $f^+ = \chi_A$ would be measurable, a contradiction. Thus f is not measurable.

4. (a) (3 points) Define a σ-algebra.

Solution: A collection $\mathcal{A}$ of subsets of a set X is a σ-algebra if whenever $E \in \mathcal{A}$ we have that $E^c \in \mathcal{A}$, and whenever E_n is a countable collection in $\mathcal{A}$, the union $\bigcup_{n=1}^{\infty} E_n$ is also in $\mathcal{A}$. Equivalently, we could phrase the second condition in terms of intersections.

(b) (5 points) Let X be a set. Call a subset $E \subseteq X$ *co-countable* if its complement E^c is countable. Let

$$\mathcal{A} = \{E \subseteq X \mid E \text{ is countable or } E \text{ is co-countable}\}.$$

Show that $\mathcal{A}$ is a σ-algebra.

Solution: If E is countable, then E^c is co-countable, and vice versa, so $\mathcal{A}$ is closed under complements. Now let E_n be a countable collection in $\mathcal{A}$. If all of the E_n are countable, then $\bigcup_{n=1}^{\infty} E_n$ is a countable union of countable sets and hence countable, so $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$. On the other hand, if E_{n_0} is co-countable for some n_0, then $E_{n_0}^c$ is countable, and so $(\bigcup_{n=1}^{\infty} E_n)^c \subseteq E_{n_0}^c$ is countable because a subset of a countable set is countable. We conclude that $\bigcup_{n=1}^{\infty} E_n$ is co-countable and hence in $\mathcal{A}$. Thus, we conclude that $\mathcal{A}$ is closed under countable unions, and so it is a σ-algebra.

5. Let $f_n \colon [a,b] \to \mathbb{R}$ be functions such that $\sup_n f_n(x)$ exists for every x. Define $g(x) = \sup_n f_n(x)$.

   (a) (3 points) Show that for all $c \in \mathbb{R}$,

   $$\{x \mid g(x) \leq c\} = \bigcap_{n=1}^{\infty} \{x \mid f_n(x) \leq c\}.$$

   Solution: If $g(x) \leq c$, then $f_n(x) \leq \sup_n f_n(x) = g(x) \leq c$ for all n, giving us the inclusion $\{x \mid g(x) \leq c\} \subseteq \bigcap_{n=1}^{\infty} \{x \mid f_n(x) \leq c\}$. Conversely, if $f_n(x) \leq c$ for all n, then c is an upper bound, so the definition of supremum tells us that $g(x) = \sup_n f_n(x) \leq c$ as well. This gives us the inclusion $\bigcap_{n=1}^{\infty} \{x \mid f_n(x) \leq c\} \subseteq \{x \mid g(x) \leq c\}$.

   (b) (3 points) Use the above fact to conclude that if f_n is measurable for all n, then g is measurable.

   Solution: We showed in class that f_n is measurable if and only if $\{x \mid f_n(x) \leq c\}$ is measurable for all c. The countable intersection of measurable sets is measurable, so we conclude that $\{x \mid g(x) \leq c\}$ is measurable for all c. That result from class then gives us that g is measurable.

6. (a) (3 points) Define almost uniform convergence.

Solution: We say that f_n converges to f almost uniformly if for all $\epsilon > 0$, there exists a measurable set E with $m(E) < \epsilon$ such that f_n converges to f uniformly on E^c.

(b) (3 points) State Egoroff's theorem for functions $[a, b] \to \mathbb{R}$.

Solution: If f_n is a sequence of measurable functions that converges to f almost everywhere, then $f_n \to f$ almost uniformly.

(c) (5 points) Consider $f_n \colon \mathbb{R} \to \mathbb{R}$ defined by $f_n = \chi_{(n,n+1)}$. Show that f_n converges to zero pointwise but f_n does not converge to zero almost uniformly.

Solution: For all x, for all $n \geq x$, we have that $f_n(x) = 0$. Thus $\lim_{n\to\infty} f_n(x) = 0$. Let $\epsilon = 1$. Assume for the sake of contradiction that $f_n \to 0$ almost uniformly. Then there exists an E such that $m(E) < 1$ and $f_n \to 0$ uniformly on E^c. In other words, $\lim_{n\to\infty} \sup_{x\in E^c} |f_n(x)| = 0$. Because $m(n, n + 1) = 1 > m(E)$, we know that $(n, n + 1)$ cannot be a subset of E. Thus, for all n, there exists an $x \in E^c$ such that $x \in (n, n + 1)$. In other words, there exists an $x \in E^c$ such that $f_n(x) = 1$. This implies that $\sup_{x\in E^c} |f_n(x)| = 1$, which does not converge to zero as $n \to \infty$. We conclude that f_n does not converge to zero almost uniformly.

7. (a) (3 points) Define what it means for a sequence of functions f_n to converge to a function f in L^1.

Solution: The sequence f_n converges to f in L^1 if $\|f - f_n\|_{L^1} := \int |f - f_n|$ converges to zero as $n \to \infty$.

(b) (3 points) Define what it means for a sequence of functions f_n to converge to a function f in measure.

Solution: Let

$$E_{n,\epsilon} = \{x \mid |f(x) - f_n(x)| \geq \epsilon\}$$

We say that f_n converges to f in measure if for all $\epsilon > 0$ we have that $\lim_{n\to\infty} m(E_{n,\epsilon}) = 0$.

(c) (5 points) Show that if f_n converges to f in L^1 then f_n converges to f in measure.

Solution: We have that

$$\int |f - f_n| \geq \int_{E_{n,\epsilon}} |f - f_n| \geq \int_{E_{n,\epsilon}} \epsilon = \epsilon\, m(E_{n,\epsilon}).$$

Then for any $\epsilon > 0$, we have that

$$m(E_{n,\epsilon}) \leq \frac{1}{\epsilon} \int |f - f_n| .$$

Since $f_n \to f$ in L^1, we know that $|f - f_n|$ converges to zero as $n \to \infty$. Thus $m(E_{n,\epsilon})$ converges to zero as $n \to \infty$ as well. We conclude that $f_n \to f$ in measure by definition.

8. (a) (3 points) Define what it means for $\phi\colon \mathbb{R} \to \mathbb{R}$ to be a step function.

Solution: The function ϕ is a step function if it is a finite linear combination of characteristic functions of finite open intervals.

Equivalently, ϕ is a step function if there exists a partition $-\infty < x_0 < x_1 < \cdots < x_n < \infty$ and values $a_1, \ldots, a_n$ such that $\phi(x) = a_k$ whenever $x \in (x_{k-1}, x_k)$ and $\phi(x) = 0$ otherwise.

(b) (3 points) Define what it means for $\phi\colon [a, b] \times [c, d] \to \mathbb{R}$ to be a step function.

Solution: The function ϕ is a step function if it is a finite linear combination of characteristic functions of rectangles, where a rectangle denotes a set of the form $(\alpha, \beta) \times (\gamma, \delta) \subseteq [a, b] \times [c, d]$.

Alternatively, ϕ is a step function if there is a partition of $[a, b] \times [c, d]$ into rectangles R_k such that ϕ is constant on the interior of R_k.

9. (8 points) State the Fubini and Tonelli theorems.

Solution: The Fubini theorem states that if $f(x, y)$ is integrable on $[a, b] \times [c, d]$, then the function $x \mapsto f(x, y)$ is integrable on $[a, b]$ for almost every y. Moreover, setting $g(y)$ to be the almost everywhere defined function $\int_a^b f(x, y)\,dx$, we have that g is integrable on $[c, d]$, and $\int g(y)\,dy = \iint f(x, y)$. We have the analogous result with the roles of x and y swapped, and we can summarize the last statement by saying that

$$\iint f(x, y) = \int_c^d \left(\int_a^b f(x, y)\,dx\right)\,dy = \int_a^b \left(\int_c^d f(x, y)\,dy\right)\,dx. \tag{1}$$

The Tonelli theorem states that if the expression

$$\int_c^d \left(\int_a^b |f(x, y)|\;dx\right)\,dy$$

is finite, then f is integrable on $[a, b] \times [c, d]$. We also have the analogous result with the roles of x and y swapped.

More precisely, the hypothesis of the Tonelli theorem is that the function $x \mapsto |f(x, y)|$ is integrable on $[a, b]$ for almost every y, and setting $g(y)$ to be the almost everywhere defined function $\int_a^b |f(x, y)|\;dx$, we have that g is integrable on $[c, d]$.

The textbook reiterates that if we can apply the Tonelli theorem then the hypothesis of the Fubini theorem holds, so we have equation (1).

10. (a) (3 points) Let $f\colon [a,b] \to \mathbb{R}$. Define the total variation of f and define what it means for f to be of bounded variation.

Solution: The total variation $V_a^b(f)$ is the supremum of the quantity

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$$

over all partitions $a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$.
If $V_a^b(f) < \infty$, we say that f is of bounded variation.

(b) (3 points) State the Jordan Decomposition Theorem.

Solution: The Jordan Decomposition Theorem states that if f is of bounded variation, then f can be written as $f = g - h$, where g and h are monotone increasing functions.

11. Let $f\colon [a,b] \to \mathbb{R}$.

   (a) (3 points) Define what it means for f to be absolutely continuous.

   Solution: A function f is absolutely continuous if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $(a_1, b_1), \ldots, (a_n, b_n)$ is a disjoint collection of intervals with total length $\sum_{k=1}^{n}(b_k - a_k)$ smaller then δ, then

   $$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon.$$

   (b) (5 points) Assume that there exists a constant M such that

   $$\frac{|f(y) - f(x)|}{|y - x|} \leq M$$

   for all $x \neq y$. Show that f is absolutely continuous.

   Solution: With notation as above, set $\delta = \frac{\epsilon}{M}$. Then

   $$\sum_{k=1}^{n} |f(b_k) - f(a_k)| \leq \sum_{k=1}^{n} M\,|b_k - a_k| = M \sum_{k=1}^{n}(b_k - a_k) < M\delta = \epsilon.$$

12. (a) (5 points) State the Fundamental Theorems of Calculus I and II in the generality given by Lebesgue.

Solution: The Fundamental Theorem of Calculus I states that if f is Lebesgue integrable on $[a, b]$, then the indefinite integral $F(x) = \int_a^x f(t)\,dt$ is absolutely continuous. Hence, F is differentiable almost everywhere, and we have that $F'(x) = f(x)$ for almost every x.

The Fundamental Theorem of Calculus II states that if f is absolutely continuous on $[a, b]$, then f is differentiable almost everywhere, the almost everywhere defined function f' is integrable on $[a, b]$, and $\int_a^x f'(t)\,dt = f(x) - f(a)$ for all x.

(b) (3 points) Give and briefly justify an example of a function $f\colon [0, 1] \to \mathbb{R}$ such that f is continuous at the endpoints 0 and 1, the function f is differentiable a.e., the almost everywhere defined function f' is integrable on $[0, 1]$, but

$$\int_0^1 f'(x)\,dx \neq f(1) - f(0).$$

Solution: One such function is $f = \chi_{[\frac{1}{2},1]}$, that is $f(x) = 0$ when $x < \frac{1}{2}$, and $f(x) = 1$ when $x \geq \frac{1}{2}$. The function f is continuous except at $x = \frac{1}{2}$. It is also differentiable except at $x = \frac{1}{2}$, and $f'(x) = 0$ whenever it exists. Thus

$$\int_0^1 f'(x)\,dx = 0 \neq 1 - 0 = f(1) - f(0).$$