Math 4121 Final

May 6, 2019

Name: ____

- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
  - Make a note on the printed page that your work continues on the back of the previous page.
  - On the back of the previous page, put a box around the work that you want graded.
- Give and use definitions from the book or from class.
- You may use any results you remember from the book or from class as long as they are more basic than the result you're asked to prove.

1. (a) (3 points) State Lebesgue's criterion for when a function is Riemann integrable.

   (b) (3 points) Give and briefly justify an example of a function that is Lebesgue integrable but not Riemann integrable.

2. (a) (3 points) State the Monotone Convergence Theorem.

(b) (5 points) Show that $f\colon [0,1] \to \mathbb{R}^*$ defined by $f(x) = \frac{1}{\sqrt{x}}$ is integrable.

(c) (5 points) Show that $f\colon [0,1] \to \mathbb{R}^*$ defined by $f(x) = \frac{1}{x}$ is not integrable.

3. (a) (3 points) Define what it means for a function $f\colon [a,b] \to \mathbb{R}^*$ to be measurable.

(b) (3 points) Show that if f is measurable then $|f|$ is measurable.

(c) (3 points) Give and justify an example of a function f such that $|f|$ is measurable but f is not measurable. You may assume the existence of a nonmeasurable subset $A \subseteq [a,b]$.

4. (a) (3 points) Define a σ-algebra.

(b) (5 points) Let X be a set. Call a subset $E \subseteq X$ *co-countable* if its complement E^c is countable. Let

$$\mathcal{A} = \{E \subseteq X \mid E \text{ is countable or } E \text{ is co-countable}\}.$$

Show that $\mathcal{A}$ is a σ-algebra.

5. Let $f_n\colon [a,b] \to \mathbb{R}$ be functions such that $\sup_n f_n(x)$ exists for every x. Define $g(x) = \sup_n f_n(x)$.

   (a) (3 points) Show that for all $c \in \mathbb{R}$,

   $$\{x \mid g(x) \leq c\} = \bigcap_{n=1}^{\infty} \{x \mid f_n(x) \leq c\}.$$

   (b) (3 points) Use the above fact to conclude that if f_n is measurable for all n, then g is measurable.

6. (a) (3 points) Define almost uniform convergence.

(b) (3 points) State Egoroff's theorem for functions $[a, b] \to \mathbb{R}$.

(c) (5 points) Consider $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n = \chi_{(n,n+1)}$. Show that f_n converges to zero pointwise but f_n does not converge to zero almost uniformly.

7. (a) (3 points) Define what it means for a sequence of functions f_n to converge to a function f in L^1.

(b) (3 points) Define what it means for a sequence of functions f_n to converge to a function f in measure.

(c) (5 points) Show that if f_n converges to f in L^1 then f_n converges to f in measure.

8. (a) (3 points) Define what it means for $\phi\colon \mathbb{R} \to \mathbb{R}$ to be a step function.

(b) (3 points) Define what it means for $\phi\colon [a, b] \times [c, d] \to \mathbb{R}$ to be a step function.

9. (8 points) State the Fubini and Tonelli theorems.

10. (a) (3 points) Let $f\colon [a,b] \to \mathbb{R}$. Define the total variation of f and define what it means for f to be of bounded variation.

(b) (3 points) State the Jordan Decomposition Theorem.

11. Let $f\colon [a,b] \to \mathbb{R}$.

    (a) (3 points) Define what it means for f to be absolutely continuous.

    (b) (5 points) Assume that there exists a constant M such that

$$\frac{|f(y) - f(x)|}{|y - x|} \leq M$$

for all $x \neq y$. Show that f is absolutely continuous.

12. (a) (5 points) State the Fundamental Theorems of Calculus I and II in the generality given by Lebesgue.

(b) (3 points) Give and briefly justify an example of a function $f\colon [0,1] \to \mathbb{R}$ such that f is continuous at the endpoints 0 and 1, the function f is differentiable a.e., the almost everywhere defined function f' is integrable on $[0,1]$, but

$$\int_0^1 f'(x)\,dx \neq f(1) - f(0).$$