Math 5052 Midterm

March 1, 2019

Name: ____

- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
  - Make a note on the printed page that your work continues on the back of the previous page.
  - On the back of the previous page, put a box around the work that you want graded.
- Give and use definitions from the book or from class.
- You may use any results you remember from the book or from class as long as they are more basic than the result you're asked to prove.

1. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed vector spaces. Let $T\colon \mathcal{X} \to \mathcal{Y}$ be a linear map.

   (a) (5 points) Define what it means for T to be bounded.

   Solution: An operator T is bounded if there exists a constant C such that

   $$\|Tx\| \leq C \|x\|$$

   for all $x \in \mathcal{X}$.

   (b) (15 points) Show that T is bounded if and only if T is continuous.

   Solution: Assume T is bounded, so there exists a C such that $\|Tx\| \leq C \|x\|$ for all $x \in X$. For all $\epsilon > 0$, set $\delta = \frac{\epsilon}{C}$. Then if $\|x - y\| < \delta$, we have that

   $$\|Tx - Ty\| = \|T(x - y)\| \leq C \|x - y\| < C\delta < \epsilon.$$

   Thus T is continuous (in fact uniformly continuous).
   Conversely, assume T is continuous. Because $B_1(0)$ is open in $\mathcal{Y}$, we know that $U := T^{-1}(B_1(0))$ is open in $\mathcal{X}$. Since $0 \in U$, we know that there exists an $\epsilon > 0$ such that $B_\epsilon(0) \subseteq U$. In other words, if $\|x\| < \epsilon$, then $\|Tx\| < 1$. Let $C = \frac{2}{\epsilon}$.
   If $x = 0$, then $\|Tx\| \leq C \|x\|$ because both sides are zero. Otherwise, let $r = \|x\|$, and let $\hat{x} = \frac{\epsilon}{2r}x$. Then $\|\hat{x}\| = \frac{\epsilon}{2} < \epsilon$, so $\|T\hat{x}\| < 1$. Multiplying both sides by $\frac{2r}{\epsilon}$ and using the linearity of T and the scaling property of the norm, we see that

   $$\|Tx\| < \frac{2r}{\epsilon} = C \|x\|,$$

   as desired.

2. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed vector spaces, and let $T\colon \mathcal{X} \to \mathcal{Y}$ be a linear map.

   (a) (5 points) Define what it means for T to be closed.

   Solution: The operator T is closed if its graph $\Gamma(T)$ is closed, where $\Gamma(T)$ is the subset of $\mathcal{X} \times \mathcal{Y}$ defined by

   $$\Gamma(T) = \{(x, Tx) \mid x \in \mathcal{X}\}.$$

   (b) (5 points) State without proof a characterization of T being closed in terms of sequences.

   Solution: The operator T is closed if whenever we have a convergent sequence $x_n \to x$ in $\mathcal{X}$ and we have $Tx_n \to y$ in $\mathcal{Y}$, then $y = Tx$.

   (c) (5 points) State the closed graph theorem.

   Solution: If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and T is closed, then T is bounded.

3. Let $\mathcal{X}$ be a Banach space and $\mathcal{Y}$ be a normed vector space.

   (a) (5 points) State the uniform boundedness principle. (The textbook provides two versions; it is okay to only state the version for a Banach space $\mathcal{X}$.)

   Solution: Let T_α be a collection of bounded linear maps between $\mathcal{X}$ and $\mathcal{Y}$. If T_α is bounded pointwise, then it is bounded uniformly.

   That is, assume that for all $x \in X$, there exists a constant C_x such that $\|T_\alpha x\| \leq C_x$ for all α. Then there exists a constant C, independent of x, such that $\|T_\alpha\| \leq C$ for all α.

   (b) (10 points) Let $T_n \in L(\mathcal{X}, \mathcal{Y})$. Assume that $\lim_{n\to\infty} T_n x$ exists for all $x \in \mathcal{X}$. Define a function $T \colon \mathcal{X} \to \mathcal{Y}$ by $Tx = \lim_{n\to\infty} T_n x$. Show that $T \in L(\mathcal{X}, \mathcal{Y})$.

   Solution: We first check that T is linear. Indeed, because addition is continuous, we know that $T_n x \to Tx$ and $T_n y \to Ty$ implies $T_n x + T_n y \to Tx + Ty$. But because T_n is linear, $T_n x + T_n y = T_n(x + y) \to T(x + y)$ by definition. We conclude that $Tx + Ty = T(x + y)$. Similarly, we know that $\lambda T_n x \to \lambda T x$. Again, because T_n is linear $\lambda T_n x = T_n(\lambda x) \to T(\lambda x)$ by definition.

   It remains to check that T is bounded. Because the norm is continuous, we know that, for all x, $\|T_n x\| \to \|Tx\|$. A convergent sequence of real numbers must be bounded, so there exists a C_x such that $\|T_n x\| \leq C_x$. We can therefore apply the uniform boundedness principle to conclude that there exists a C such that $\|T_n\| \leq C$ for all n.

   In other words, for all $x \in \mathcal{X}$, we have that $\|T_n x\| \leq C \|x\|$. Because $T_n x \to Tx$ and the norm is continuous, we conclude that $\|Tx\| \leq C \|x\|$, as desired.

4. Let l^1 denote $L^1(\mathbb{N})$ with respect to counting measure, equipped with the L^1 norm. That is, l^1 consists of sequences $x\colon \mathbb{N} \to \mathbb{R}$ such that $\|x\|_{l^1} := \sum_{k=1}^{\infty} |x(k)| < \infty$.

   Let l^∞ denote $B(\mathbb{N})$ equipped with the uniform norm. That is, l^∞ consists of sequences such that $\|y\|_{l^\infty} := \sup_k |y(k)| < \infty$.

   (a) (10 points) Let $f \in (l^1)^*$. Show that there exists a $y \in l^\infty$ such that for all $x \in l^1$, we have

$$f(x) = \sum_{k=1}^{\infty} y(k)x(k). \tag{1}$$

   (This is the key step in the proof that $(l^1)^* = l^\infty$, a fact that you may use on the other problems of this exam.)

Solution:
Define e_n as in the next problem, and let $y(k) = f(e_k)$. Note that

$$|y(k)| = |f(e_k)| \leq \|f\|_{(l^1)^*} \|e_k\|_{l^1} = \|f\|_{(l^1)^*},$$

so $y \in l^\infty$. It remains to check that f acts on l^1 via the formula (1). Let $x \in l^1$, and let x_n be the truncation of x, that is, $x_n(k) = x(k)$ for $k \leq n$ and $x_n(k) = 0$ for $k > n$. Then $x_n = \sum_{k=1}^{n} x(k)e_k$. Because f is linear, it respects finite linear combinations, so

$$f(x_n) = \sum_{k=1}^{n} x(k)f(e_k) = \sum_{k=1}^{n} y(k)x(k).$$

But we have also seen in the homework that $x_n \to x$ in l^1. Indeed,

$$\|x - x_n\|_{l^1} = \sum_{k=n+1}^{\infty} |x(k)| \to 0$$

as $n \to \infty$. Thus, because f is continuous, we see that

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sum_{k=1}^{n} y(k)x(k) = \sum_{k=1}^{\infty} y(k)x(k),$$

as desired.

We've seen $e_n\colon \mathbb{N} \to \mathbb{R}$ defined by $e_n(n) = 1$ and $e_n(k) = 0$ for $k \neq n$.

   (b) (10 points) Show that $\{e_n\}_{n=1}^{\infty}$ does not converge with respect to the weak topology on l^1.

Solution: If the sequence e_n converged weakly to some $x \in l^1$, that would mean by definition that $f(e_n) \to f(x)$ for all $f \in (l^1)^*$. Thus, to show that e_n does not converge, it suffices to find an $f \in (l^1)^*$ such that $f(e_n)$ does not converge. Let

$y(k) = (-1)^k$, which is clearly in l^∞, and define $f(x) = \sum y(k)x(k)$ as before. Then $f(e_n) = y(n) = (-1)^n$, which does not converge.

(c) (10 points) Recall from class that $l^1 = (c_0)^*$, where c_0 denotes the set of sequences $y\colon \mathbb{N} \to \mathbb{R}$ such that $\lim_{k\to\infty} y(k) = 0$, equipped with the uniform norm. In this context, prove that $\{e_n\}_{n=1}^\infty$ converges with respect to the weak* topology on l^1.

Solution:
I claim that e_n converges to zero in the weak* topology. Given any $y \in c_0$, let $\hat{y}$ be the corresponding functional in $(c_0)^{**} = (l^1)^*$, that is, $\hat{y}(x) = \sum y(k)x(k)$ for all $x \in l^1$. To show that e_n converges to zero in the weak* topology, we must show that $\hat{y}(e_n)$ converges to zero for all y. But $\hat{y}(e_n) = y(n)$, which converges to zero by definition of c_0. Thus, e_n converges to 0 in the weak* topology.

5. Let c denote the set of sequences $y\colon \mathbb{N} \to \mathbb{R}$ such that $\lim_{k\to\infty} y(k)$ exists, equipped with the uniform norm. (The c stands for convergent. Compare to c_0, the sequences converge to 0.) Recall the definition of l^∞ from the previous problem.

   (a) (5 points) Let $f\colon c \to \mathbb{R}$ be defined by $f(y) = \lim_{k\to\infty} y(k)$. Show that f is a bounded linear functional on c. (You may cite any facts you know about sequences of real numbers without proof.)

   Solution: Properties of limits tell us that f is linear in x. It remains to show that f is bounded, and there are many ways to do so. For example, because the absolute value is continuous, we know that

   $$|f(y)| = \lim_{k\to\infty} |y(k)| = \limsup_{k\to\infty} |y(k)| \le \sup_k |y(k)| = \|y\|_{c_0} = \|y\|_{l^\infty}.$$

   (b) (5 points) Use an important result from class to show that there is a bounded linear functional $\tilde{f}\colon l^\infty \to \mathbb{R}$ such that the restriction of $\tilde{f}$ to c is f. (Such a functional $\tilde{f}$ lets us define the "limit" of any bounded sequence, whether or not it converges!)

   Solution: We are using the uniform norm on both c_0 and l^∞, so c_0 is a subspace of l^∞ as normed vector spaces. The Hahn-Banach Theorem tells us that because $|f(y)| \le \|y\|_{l^\infty}$ for all $y \in c_0$, it is possible to extend f to a linear functional $\tilde{f}$ on all of l^∞ such that $\left|\tilde{f}(y)\right| \le \|y\|_{l^\infty}$ for all $y \in l^\infty$, as desired.

   (c) (10 points) Recall that there is a natural inclusion $l^1 \subseteq (l^1)^{**} = (l^\infty)^*$. Show that $\tilde{f} \notin l^1$, and conclude that l^1 is not equal to $(l^1)^{**}$.

   Solution: Define e_n as before, but this time we think of e_n as elements of c. Because $e_n(k) = 0$ except for $k = n$, we know that $f(e_n) = \lim_{k\to\infty} e_n(k) = 0$ for all n. Thus $\tilde{f}(e_n) = 0$ for all n as well.
   Assume for the sake of contradiction that $\tilde{f} \in l^1$, which means that there is a sequence $x \in l^1$ such that $\tilde{f}(y) = \hat{x}(y) = \sum_{k=1}^{\infty} y(k)x(k)$ for all $y \in l^\infty$. In particular, $\tilde{f}(e_n) = x(n)$. But we saw that $\tilde{f}(e_n) = 0$ for all n, so $x = 0$, and so $\tilde{f} = 0$. But we know that $\tilde{f}$ is not 0. Take any sequence with a nonzero limit, such as $y(k) \equiv 1$. Then $\tilde{f}(y) = f(y) = 1 \ne 0$. We have arrived at a contradicicition, so $\tilde{f} \notin l^1$, as desired.