Math 5052 Midterm

March 1, 2019

Name: ____

- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
  - Make a note on the printed page that your work continues on the back of the previous page.
  - On the back of the previous page, put a box around the work that you want graded.
- Give and use definitions from the book or from class.
- You may use any results you remember from the book or from class as long as they are more basic than the result you're asked to prove.

1. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed vector spaces. Let $T\colon \mathcal{X} \to \mathcal{Y}$ be a linear map.
   (a) (5 points) Define what it means for T to be bounded.

   (b) (15 points) Show that T is bounded if and only if T is continuous.

2. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed vector spaces, and let $T \colon \mathcal{X} \to \mathcal{Y}$ be a linear map.

   (a) (5 points) Define what it means for T to be closed.

   (b) (5 points) State without proof a characterization of T being closed in terms of sequences.

   (c) (5 points) State the closed graph theorem.

3. Let $\mathcal{X}$ be a Banach space and $\mathcal{Y}$ be a normed vector space.
   (a) (5 points) State the uniform boundedness principle. (The textbook provides two versions; it is okay to only state the version for a Banach space $\mathcal{X}$.)

   (b) (10 points) Let $T_n \in L(\mathcal{X}, \mathcal{Y})$. Assume that $\lim_{n\to\infty} T_n x$ exists for all $x \in \mathcal{X}$. Define a function $T\colon \mathcal{X} \to \mathcal{Y}$ by $Tx = \lim_{n\to\infty} T_n x$. Show that $T \in L(\mathcal{X}, \mathcal{Y})$.

4. Let l^1 denote $L^1(\mathbb{N})$ with respect to counting measure, equipped with the L^1 norm. That is, l^1 consists of sequences $x\colon \mathbb{N} \to \mathbb{R}$ such that $\|x\|_{l^1} := \sum_{k=1}^{\infty} |x(k)| < \infty$.

   Let l^∞ denote $B(\mathbb{N})$ equipped with the uniform norm. That is, l^∞ consists of sequences such that $\|y\|_{l^\infty} := \sup_k |y(k)| < \infty$.

   (a) (10 points) Let $f \in (l^1)^*$. Show that there exists a $y \in l^\infty$ such that for all $x \in l^1$, we have

   $$f(x) = \sum_{k=1}^{\infty} y(k)x(k). \tag{1}$$

   (This is the key step in the proof that $(l^1)^* = l^\infty$, a fact that you may use on the other problems of this exam.)

   We've seen $e_n\colon \mathbb{N} \to \mathbb{R}$ defined by $e_n(n) = 1$ and $e_n(k) = 0$ for $k \neq n$.

   (b) (10 points) Show that $\{e_n\}_{n=1}^{\infty}$ does not converge with respect to the weak topology on l^1.

   (c) (10 points) Recall from class that $l^1 = (c_0)^*$, where c_0 denotes the set of sequences $y\colon \mathbb{N} \to \mathbb{R}$ such that $\lim_{k\to\infty} y(k) = 0$, equipped with the uniform norm. In this context, prove that $\{e_n\}_{n=1}^{\infty}$ converges with respect to the weak* topology on l^1.

5. Let c denote the set of sequences $y\colon \mathbb{N} \to \mathbb{R}$ such that $\lim_{k\to\infty} y(k)$ exists, equipped with the uniform norm. (The c stands for convergent. Compare to c_0, the sequences converge to 0.) Recall the definition of l^∞ from the previous problem.

   (a) (5 points) Let $f\colon c \to \mathbb{R}$ be defined by $f(y) = \lim_{k\to\infty} y(k)$. Show that f is a bounded linear functional on c. (You may cite any facts you know about sequences of real numbers without proof.)

   (b) (5 points) Use an important result from class to show that there is a bounded linear functional $\tilde{f}\colon l^\infty \to \mathbb{R}$ such that the restriction of $\tilde{f}$ to c is f. (Such a functional $\tilde{f}$ lets us define the "limit" of any bounded sequence, whether or not it converges!)

   (c) (10 points) Recall that there is a natural inclusion $l^1 \subseteq (l^1)^{**} = (l^\infty)^*$. Show that $\tilde{f} \notin l^1$, and conclude that l^1 is not equal to $(l^1)^{**}$.