Math 5052 Final

May 3, 2019

Name: ____

- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
  - Make a note on the printed page that your work continues on the back of the previous page.
  - On the back of the previous page, put a box around the work that you want graded.
- Give and use definitions from the book or from class.
- You may use any results you remember from the book or from class as long as they are more basic than the result you're asked to prove.

1. (5 points) Let $\mathcal{H}$ be a Hilbert space. Define what it means to be an orthonormal basis for $\mathcal{H}$.

2. (a) (5 points) State the Riesz Representation Theorem for Hilbert spaces.

(b) (10 points) Let $\mathcal{H}$ be a Hilbert space, and let $T\colon \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. Show that there is a unique bounded linear operator $T^*\colon \mathcal{H} \to \mathcal{H}$ such that $\langle Tx, y\rangle = \langle x, T^* y\rangle$ for all $x, y \in \mathcal{H}$.

3. (5 points) Let $\mathcal{H}$ be a Hilbert space, and let $T\colon \mathcal{H} \to \mathcal{H}$ be a linear map. Assume that $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in \mathcal{H}$. Prove that T is bounded.

   Hint: Closed Graph Theorem.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $1 \leqslant p \leqslant \infty$.

   (a) (5 points) Define $L^p(X)$, including $L^\infty(X)$.

   (b) (5 points) State Hölder's inequality.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $1 \leqslant p \leqslant \infty$.

   (a) (5 points) If μ is finite, show that $L^p(X) \subseteq L^1(X)$ and that the inclusion map is continuous.

   (b) (5 points) If μ is counting measure, show that $L^p(X) \subseteq L^\infty(X)$ and that the inclusion map is continuous.

   (c) (5 points) Give and justify a counterexample to (a), in the following sense. Write down a measure space $(X, \mathcal{M}, \mu)$ such that for any $p > 1$, there exists a function f such that $f \in L^p(X)$ but $f \notin L^1(X)$.

   (d) (5 points) Give and justify a counterexample to (b), in the same sense as above. Write down a measure space $(X, \mathcal{M}, \mu)$ such that for any $p < \infty$, there exists a function f such that $f \in L^p(X)$ but $f \notin L^\infty(X)$.

6. Consider $\mathbb{N}$ with counting measure. Let $f_n = \chi_{\{n\}}$, that is, $f_n(n) = 1$, and $f_n(k) = 0$ otherwise.

   (a) (5 points) For $1 \leqslant p \leqslant \infty$, show that f_n does *not* converge to zero in l^p (with respect to the norm topology).

   (b) (5 points) For $1 < p < \infty$, show that f_n converges to zero weakly in l^p.

   (c) (5 points) Show that f_n does *not* converge to zero weakly in l^1.

   (d) (5 points) Show that f_n converges to zero weakly in l^∞. Hint: consider $\pm f_1 \pm \cdots \pm f_n$.

7. (5 points) As in class: Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and let $s(\mathbb{Z})$ denote the space of sequences that decay faster than any polynomial.

   (a) State the definition of the Fourier transform $C^\infty(\mathbb{T}) \to s(\mathbb{Z})$.

   (b) State the definition of the inverse Fourier transform $s(\mathbb{Z}) \to C^\infty(\mathbb{T})$.

8. As in class: If $f\colon C^\infty(\mathbb{T}) \to \mathbb{C}$ is a continuous linear functional, denote f applied to ϕ with the notation $\int_{\mathbb{T}} f\phi$ or the equivalent notation $\langle f, \bar{\phi}\rangle_{\mathbb{T}}$. Let $\mathcal{D}(\mathbb{T})$ denote the space of all such f. Let $s'(\mathbb{Z})$ denote $(s(\mathbb{Z}))^*$, the space of all continuous linear functionals on $s(\mathbb{Z})$. As before, we denote $a \in s'(\mathbb{Z})$ applied to $\sigma \in s(\mathbb{Z})$ by $\sum_\kappa a\sigma$ or by $\langle a, \bar{\sigma}\rangle_{\mathbb{Z}}$.

   (a) (5 points) Let $f \in \mathcal{D}(\mathbb{T})$. Define the distributional derivative $\frac{d}{dx} f$, or, equivalently, define Df, where $D = \frac{1}{2\pi i}\frac{d}{dx}$.

   (b) (5 points) Show that $\widehat{Df} = \kappa \hat{f}$.

   (c) (5 points) In class, we showed that if $a \in s'(\mathbb{Z})$, then $\frac{a}{1+|\kappa|^\alpha} \in l^2(\mathbb{Z})$ for some sufficiently large nonnegative integer α. An equivalent way of framing this result is that if $a \in s'(\mathbb{Z})$ and $a(0) = 0$, then $a = \kappa^\alpha b$ for some $b \in l^2(\mathbb{Z})$. You may use this result without proof.

   If $f \in \mathcal{D}$, show that $f = D^\alpha g + C$ for some nonnegative integer α, some $g \in L^2(\mathbb{T})$, and some $C \in \mathbb{C}$. Show furthermore that if $\alpha \geqslant 1$ then C must be $\int_{\mathbb{T}} f$, that is, $\langle f, 1\rangle_{\mathbb{T}}$.

9. (5 points) Let $f \in \mathcal{D}(\mathbb{T})$ and $\phi \in C^{\infty}(\mathbb{T})$. State the definition of the convolution $f * \phi$.