Math 603 Midterm

November 8, 2019

1. State the mean-value property and an associated theorem.

> **Solution:** Let U be an open bounded set. A function $u\colon U \to \mathbb{R}$ has the mean value property if
> $$u(x) = \fint_{\partial B(x,r)} u(y)\, dy = \fint_{B(x,r)} u(y)\, dy$$
> for all $B(x,r) \subset U$.
>
> We have a theorem that if $u \in C^2(U)$ is harmonic, then it has the mean value property.
>
> Alternatively, we have a theorem that if $u \in C(U)$ has the mean value property, then it is harmonic.

2. (a) State the maximum principle.

> **Solution:** Let U be an open bounded set. The maximum principle states that if $u \in C^2(U) \cap C(\overline{U})$ is harmonic, then the maximum value of u occurs on the boundary, that is, $\max_{\overline{U}} u = \max_{\partial U} u$.

(b) State the strong maximum principle.

> **Solution:** Let U be a connected open bounded set. The strong maximum principle states that if $u \in C^2(U) \cap C(\overline{U})$ is harmonic, if u attains its maximum on the interior, then u is constant. More precisely, if there exists an $x_0 \in U$ such that $u(x_0) = \max_{\overline{U}} u$, then u is constant.

3. Let U be an open bounded domain, and assume that there is a continuous function $K : U \times \partial U \to \mathbb{R}$ with the following property:

- For all $g \in C(\partial U)$, we can define a function $u\colon U \to \mathbb{R}$ by
$$u(x) = \int_{\partial U} K(x,y) g(y)\, dy.$$
This function u is harmonic on U, and u can be extended to a continuous function on $\overline{U}$ that is equal to g on ∂U.

Use a result from class to show that $K(x,y) \geq 0$ for all $x \in U$ and $y \in \partial U$.

Solution: We showed in class that one of the consequences of the maximum principle is that if u is harmonic and its boundary value is nonnegative, then u is nonnegative. Intuitively, then only way for u to be nonnegative whenever g is nonnegative in the above formula is for K to be nonnegative.

To prove this fact, assume for the sake of contradiction that $K(x_0, y_0) < 0$. Because K is continuous, it must be negative in a neighborhood of this point. More precisely, there is a δ such that $K(x_0, y) < 0$ whenever $|y - y_0| < \delta$. Set $g\colon \partial U \to \mathbb{R}$ to be a continuous function that is positive on this neighborhood and zero outside of this neighborhood. For example, we could set $g(y)$ to be $\delta - |y - y_0|$ whenever y is in this neighborhood, and $g(y)$ to be 0 otherwise. Then

$$u(x_0) = \int_{\partial U \cap B^0(y_0,\delta)} K(x_0, y) g(y)\, dy + \int_{\partial U \setminus B^0(y_0,\delta)} K(x_0, y) g(y)\, dy.$$

The second integral is zero because $g(y) = 0$ for y outside this neighborhood. Meanwhile, the first integral is negative because $K(x,y) < 0$ and $g(y) > 0$ on $\partial U \cap B^0(y_0, \delta)$. We conclude that $u(x_0) < 0$, a contradiction.

4. Let $f \in C(\overline{U})$. For any function $w \in C^2(\overline{U})$, define

$$L(w) = \int_U \left(\tfrac{1}{2} |\nabla w|^2 - f w \right) dx.$$

Let $u \in C^2(\overline{U})$, and assume that $L(u) \leq L(w)$ for all $w \in C^2(\overline{U})$ such that $w = u$ on ∂U. Prove that $-\Delta u = f$.

Solution: Let $v \in C^2(\overline{U})$ be an arbitrary function such that $v = 0$ on ∂U. Let $w_s = u + sv$. Then $w_s \in C^2(\overline{U})$ and its value on ∂U is $u + s \cdot 0 = u$. Since $w_0 = u$, we have by the hypothesis that $L(w_0) \leq L(w_s)$ for all s. Thus, the function $s \mapsto L(w_s)$ has a minimum at $s = 0$, from which we conclude that

$$\begin{aligned} 0 &= \left.\frac{d}{ds}\right|_{s=0} L(w_s), \\ &= \int_U \left.\frac{d}{ds}\right|_{s=0} \left(\tfrac{1}{2} |\nabla(u+sv)|^2 - f(u+sv) \right) dx, \\ &= \int_U \left.\frac{d}{ds}\right|_{s=0} \left(\tfrac{1}{2} |\nabla u|^2 + s \nabla u \cdot \nabla v + \tfrac{s^2}{2} |\nabla v|^2 - fu - sfv \right) dx, \\ &= \int_U \left(\nabla u \cdot \nabla v - fv \right) dx, \\ &= \int_U \left(-\Delta u - f \right) v \, dx, \end{aligned}$$

by integration by parts because the boundary term $\int_{\partial U} v \nabla u \cdot n$ vanishes because $v = 0$ on ∂U.

Since $\int_U (-\Delta u - f) v \, dx = 0$ for all v that vanish on the boundary, we conclude by the fundamental lemma of variational calculus that $-\Delta u - f = 0$, as desired.