Math 603 Homework 4

1. (Evans, Problem 2.5.5) We say $v \in C^2(\overline{U})$ is *subharmonic* if

$$-\Delta v \leq 0 \quad \text{in } U.$$

   (a) Prove for subharmonic v that

$$v(x) \leq \fint_{B(x,r)} v \, \mathrm{d}y \quad \text{for all } B(x,r) \subset U.$$

   (b) Prove that therefore $\max_{\overline{U}} v = \max_{\partial U} v$.

   (c) Let $\phi \colon \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.

   (d) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic.

2. (Evans, Problem 2.5.6) Let U be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C, depending only on U, such that

$$\max_{\overline{U}} |u| \leq C\big(\max_{\partial U} |g| + \max_{\overline{U}} |f|\big),$$

   whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$$

   (Hint: $-\Delta(u + \frac{|x|^2}{2n}\lambda) \leq 0$, for $\lambda := \max_{\overline{U}} |f|$.)

3. (Evans, Problem 2.5.7) Use Poisson's formula for the ball to prove

$$r^{n-2} \frac{r - |x|}{\big(r + |x|\big)^{n-1}} u(0) \leq u(x) \leq r^{n-2} \frac{r + |x|}{\big(r - |x|\big)^{n-1}} u(0),$$

   whenever u is positive and harmonic in $B^0(0,r)$. This is an explicit form of Harnack's inequality.

   Warning: you only have information about u on $B^0(0,r)$, not $B(0,r)$. It's quite possible that u approaches infinity as u gets closer to $\partial B(0,r)$. Thus, you can't apply Poisson's formula to $B(0,r)$, since the boundary value might not even exist. Instead, carefully choose a smaller radius, and at the end take the limit as this radius approaches r.

4. (Evans, Problem 2.5.10)

(a) Let U^+ denote the open half-ball $\{x \in \mathbb{R}^n : |x| < 1,\ x_n > 0\}$. Assume $u \in C^2(\overline{U^+})$ is harmonic in U^+, with $u = 0$ on $\partial U^+ \cap \{x_n = 0\}$. Set

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, \dots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

for $x \in U = B^0(0,1)$. Prove $v \in C^2(U)$ and thus v is harmonic within U.

(b) Now assume only that $u \in C^2(U^+) \cap C(\overline{U^+})$. Show that v is harmonic within U. (Hint: Use Poisson's formula for the ball.)