Math 603 Homework 3

1. (Evans, Problem 2.5.3) Modify the proof of the mean-value formulas to show for $n \geq 3$ that

$$u(0) = \fint_{\partial B(0,r)} g \, \mathrm{d}S + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f \, \mathrm{d}x,$$

   provided

$$\begin{cases} -\Delta u = f & \text{in } B^0(0,r) \\ u = g & \text{on } \partial B(0,r). \end{cases}$$

2. (Evans, Problem 2.5.4) Give a direct proof that if $u \in C^2(U) \cap C(\overline{U})$ is harmonic within a bounded open set U, then

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

   (Hint: Define $u_\epsilon := u + \epsilon |x|^2$ for $\epsilon > 0$, and show u_ϵ cannot attain its maximum over $\overline{U}$ at an interior point.)

   More specifically, when Evans says "direct," he means that you shouldn't prove this using the mean value theorems, which is what we did in class.

3. (Evans, Problem 2.5.9) Let u be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+, \\ u = g & \text{on } \partial \mathbb{R}^n_+, \end{cases}$$

   given by Poisson's formula for the half-space. Assume g is bounded and $g(x) = |x|$ for $x \in \partial \mathbb{R}^n_+$, $|x| \leq 1$. Show Du is *not* bounded near $x = 0$. (Hint: Estimate $\frac{u(\lambda e_n) - u(0)}{\lambda}$.)

   Note: See Appendix A.2 for the meaning of the notation e_n.