Math 603 Homework 2

1. Evans, Problem 2.5.11

   (Kelvin transform for the Laplace equation) The *Kelvin transform* $\mathcal{K}u = \bar{u}$ of a function $u\colon \mathbb{R}^n \to \mathbb{R}$ is

   $$\bar{u}(x) := u(\bar{x})\,|\bar{x}|^{n-2} = u(x/\,|x|^2)\,|x|^{2-n} \qquad (x \neq 0),$$

   where $\bar{x} = x/\,|x|^2$. Show that if u is harmonic, then so is $\bar{u}$.

   (Hint: First show that $D_x\bar{x}(D_x\bar{x})^T = |\bar{x}|^4\, I$. The mapping $x \mapsto \bar{x}$ is *conformal*, meaning angle preserving.)

   Yasha's note: I also found it helpful to prove that $\Delta(vw) = (\Delta v)w + 2\nabla v \cdot \nabla w + v(\Delta w)$, as well as to compute $\Delta \bar{x}_i$, though I'm sure there are many other ways to do this computation.