Math 603 Final

December 20, 2019

- Please respond to parts A and B on separate sheets of paper.
- Do any seven of the eight problems on the exam. Clearly state which problem number you are skipping at the top of your solutions to part B.
- Write your name on every sheet of paper you are submitting.
- Give and use definitions from the book or from class.
- You may use any results you remember from the book or from class as long as they are more basic than the result you're asked to prove.

Part A

1. State the Harnack's inequality theorem.

Solution: Let U be a open bounded set, and let $V \subset\subset U$ be a connected open set. There exists a constant C depending only on U and V such that for any nonnegative harmonic function $u \colon U \to \mathbb{R}$, we have

$$\frac{1}{C} u(x) \leq u(y) \leq C u(x)$$

for all $x, y \in V$.

2. Let U be an open bounded domain, and let $f \colon U \to \mathbb{R}$ and $g \colon \partial U \to \mathbb{R}$. Show that there is at most one solution in $C^2(U) \cap C(\overline{U})$ to

$$\begin{aligned} -\Delta u &= f \text{ on } U, \\ u &= g \text{ on } \partial U. \end{aligned}$$

Solution: If u and $\tilde{u}$ are two such solutions, then let $v = u - \tilde{u}$, so $v \in C^2(U) \cap C(\overline{U})$ satisfies

$$\begin{aligned} -\Delta v &= 0 \text{ on } U, \\ v &= 0 \text{ on } \partial U. \end{aligned}$$

Since v is harmonic, by the maximum principle, the maximum value of v occurs on the boundary. Since $v = 0$ on the boundary, we conclude that $v \le 0$ on U. Likewise, the minimum value of v occurs on the boundary, so $v \ge 0$. We conclude that $v = 0$, so $u = \tilde{u}$.

Alternatively, we can use energy methods. We integrate by parts to see that

$$\int_U |\nabla v|^2 \, dx = \int_U \nabla v \cdot \nabla v \, dx = \int_U v(-\Delta v) \, dx + \int_{\partial U} v \nabla v \cdot n \, dx = 0$$

because $\Delta v = 0$ on U and $v = 0$ on ∂U. We conclude that $\nabla v = 0$ on U, so v is constant. Because $v = 0$ on ∂U, we conclude that this constant is zero, so $u = \tilde{u}$.

Part B

3. Let $U \subset \mathbb{R}^n$ be open.

(a) State what it means for a function $u : U \to \mathbb{R}$ to be weakly differentiable with respect to x_i ($1 \le i \le n$).

Solution: A function $u \in L^1_{\text{loc}}(U)$ is weakly differentiable with respect to x_i if there exists a function $v \in L^1_{\text{loc}}(U)$ such that

$$\int_U v\phi \, dx = -\int_U u \frac{\partial \phi}{\partial x_i} \, dx, \quad \forall \phi \in C_c^\infty(U).$$

(b) Show that the function $u(x) = 1 - |x|$ is weakly differentiable on $U = (-1, 1)$.

Solution:

$$\begin{aligned} \int_{-1}^1 u \frac{\partial \phi}{\partial x} \, dx &= \int_{-1}^0 u \frac{\partial \phi}{\partial x} \, dx + \int_0^1 u \frac{\partial \phi}{\partial x} \, dx \\ &= u\phi\big|_{-1}^0 - \int_{-1}^0 (1)\phi \, dx + u\phi\big|_0^1 - \int_0^1 (-1)\phi \, dx \end{aligned}$$

since $\frac{\partial u}{\partial x} = 1$ on $(-1, 0)$ and $\frac{\partial u}{\partial x} = -1$ on $(0, 1)$. Since $\phi(-1) = \phi(1) = 0$, and since $u\phi$ is continuous at $x = 0$, we have $u\phi\big|_{-1}^0 + u\phi\big|_0^1 = 0$. Thus,

$$\int_{-1}^1 u \frac{\partial \phi}{\partial x} \, dx = -\int_{-1}^1 v\phi \, dx,$$

where

$$v(x) = \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{if } x > 0. \end{cases}$$

Since $v \in L^1_{\text{loc}}(U)$, u is weakly differentiable.

4. State the definition of the Sobolev spaces $W^{k,p}(U)$ and the Sobolev norms $\|\cdot\|_{W^{k,p}(U)}$ for $k \in \mathbb{N}_0$ and $p \in [1, \infty]$.

Solution: We have

$$W^{k,p}(U) = \{u \in L^p(U) \mid \forall |\alpha| \leq k, \text{ the weak derivative } D^\alpha u \text{ exists and belongs to } L^p(U)\},$$

and

$$\|u\|_{W^{k,p}(U)} = \begin{cases} \left(\sum_{|\alpha|\leq k} \|D^\alpha u\|^p_{L^p(U)}\right)^{1/p}, & \text{if } p \in [1, \infty), \\ \max_{|\alpha|\leq k} \|D^\alpha u\|_{L^\infty(U)}, & \text{if } p = \infty. \end{cases}$$

5. Let $U \subset \mathbb{R}^n$ be open and bounded with C^1 boundary, and let $u \in W^{1,1}(U)$. Use the trace inequality to show that if there exists a sequence $\{u_j\}_{j=1}^\infty \subset C_c^\infty(U)$ satisfying $\lim_{j\to\infty} \|u_j - u\|_{W^{1,1}(U)} = 0$, then the trace of u on ∂U is 0.

Solution: For every j, we have

$$\|Tu\|_{L^1(\partial U)} \leq \|Tu - Tu_j\|_{L^1(\partial U)} + \|Tu_j\|_{L^1(\partial U)}.$$

The second term vanishes since u_j has compact support. By the trace inequality, the first term satisfies

$$\|Tu - Tu_j\|_{L^1(\partial U)} = \|T(u - u_j)\|_{L^1(\partial U)} \leq C\|u - u_j\|_{W^{1,1}(U)} \to 0, \quad \text{as } j \to \infty.$$

Thus, $Tu = 0$.

6. Let $U \subset \mathbb{R}^n$ be open, and let $u \in L^2(U)$. Show that $u \in H^1(U)$ if and only if there exists a constant $C > 0$ such that for each $j = 1, 2, \ldots, n$,

$$\left| \int_U u \frac{\partial \phi}{\partial x_j} \, dx \right| \leq C\|\phi\|_{L^2(U)}, \quad \forall \phi \in C_c^\infty(U). \tag{1}$$

Solution: If $u \in H^1(U)$, then

$$\int_U u \frac{\partial \phi}{\partial x_j}\, dx = -\int_U \frac{\partial u}{\partial x_j} \phi\, dx, \quad \forall \phi \in C_c^\infty(U),$$

where $\frac{\partial u}{\partial x_j} \in L^2(U)$ denotes the weak derivative of u with respect to x_j. By the Cauchy-Schwarz inequality,

$$\left| -\int_U \frac{\partial u}{\partial x_j} \phi\, dx \right| \le C \|\phi\|_{L^2(U)}, \quad \forall \phi \in C_c^\infty(U),\ j = 1, 2, \dots, n,$$

where $C = \max_j \left\| \frac{\partial u}{\partial x_j} \right\|_{L^2(U)}$.

Conversely, suppose (1) holds. Then for each j, we can define a map $\ell : L^2(U) \to \mathbb{R}$ by

$$\ell(v) = \lim_{k\to\infty} \int_U u \frac{\partial v_k}{\partial x_j}\, dx,$$

where $\{v_k\}_{k=1}^\infty \subset C_c^\infty(U)$ is any sequence satisfying $\|v - v_k\|_{L^2(U)} \to 0$. It is easy to check that ℓ is well-defined (the limit exists and is independent of the choice of $\{v_k\}_{k=1}^\infty$), bounded, and linear, so the Riesz representation theorem ensures the existence of $w \in L^2(U)$ such that

$$\ell(v) = \int_U wv\, dx, \quad \forall v \in L^2(U).$$

In particular,

$$\int_U u \frac{\partial \phi}{\partial x_j}\, dx = \int_U w\phi\, dx, \quad \forall \phi \in C_c^\infty(U).$$

This shows that u is weakly differentiable with respect to x_j, and $\frac{\partial u}{\partial x_j} = -w \in L^2(U)$. Hence $u \in H^1(U)$.

7. Let $U \subset \mathbb{R}^n$ be open and bounded, and let $f \in L^2(U)$. Write down a weak formulation of the problem

$$-\Delta u + u = f, \text{ in } U,$$
$$u = 0, \text{ on } \partial U,$$

and show that it has a unique weak solution.

Solution: A weak solution of the above problem is a function $u \in H_0^1(U)$ satisfying

$$a(u, v) = \ell(v), \quad \forall v \in H_0^1(U),$$

where $a(u, v) = \int_U (Du \cdot Dv + uv)\, dx$ and $\ell(v) = \int_U fv\, dx$. Let us check the hypotheses of the Lax-Milgram theorem. Clearly a is bilinear and ℓ is linear. Since $a(u, v)$ is just the $H^1(U)$-inner product, we have

$$|a(u, v)| = |(u, v)_{H^1(U)}| \le \|u\|_{H^1(U)} \|v\|_{H^1(U)}, \quad \forall u, v \in H_0^1(U),$$

and

$$a(u,u) = (u,u)_{H^1(U)} = \|u\|^2_{H^1(U)}, \quad \forall u \in H^1_0(U).$$

Furthermore,

$$|\ell(v)| \leq \|f\|_{L^2(U)} \|v\|_{L^2(U)} \leq \|f\|_{L^2(U)} \|v\|_{H^1(U)}, \quad \forall v \in H^1_0(U).$$

By the Lax-Milgram theorem, there exists a unique function $u \in H^1_0(U)$ satisfying $a(u,v) = \ell(v)$ for all $v \in H^1_0(U)$.

8. Let $U \subset \mathbb{R}^n$ be open and bounded with C^1 boundary. Show that if $n = 3$ and $f \in L^{6/5}(U)$, then the map

$$\begin{aligned} \ell : H^1(U) &\to \mathbb{R} \\ v &\mapsto \int_U fv\,dx \end{aligned}$$

is a bounded linear functional on $H^1(U)$. (Hint: Use a Sobolev inequality.)

Solution: Clearly ℓ is linear. By Holder's inequality,

$$|\ell(v)| \leq \|f\|_{L^{6/5}(U)} \|v\|_{L^6(U)}.$$

Applying the Sobolev inequality $\|v\|_{L^{p^*}(U)} \leq C\|v\|_{W^{1,p}(U)}$ with $p = 2$, $n = 3$, and $p^* = \frac{np}{n-p} = 6$ gives

$$|\ell(v)| \leq C\|f\|_{L^{6/5}(U)} \|v\|_{W^{1,2}(U)} = C\|f\|_{L^{6/5}(U)} \|v\|_{H^1(U)}.$$

Thus, ℓ is bounded on $H^1(U)$.