Math 603 Final

December 20, 2019

- Please respond to parts A and B on separate sheets of paper.
- Do any seven of the eight problems on the exam. Clearly state which problem number you are skipping at the top of your solutions to part B.
- Write your name on every sheet of paper you are submitting.
- Give and use definitions from the book or from class.
- You may use any results you remember from the book or from class as long as they are more basic than the result you're asked to prove.

Part A

1. State the Harnack's inequality theorem.
2. Let U be an open bounded domain, and let $f\colon U \to \mathbb{R}$ and $g\colon \partial U \to \mathbb{R}$. Show that there is at most one solution in $C^2(U) \cap C(\overline{U})$ to

$$\begin{aligned} -\Delta u &= f \text{ on } U, \\ u &= g \text{ on } \partial U. \end{aligned}$$

Part B

3. Let $U \subset \mathbb{R}^n$ be open.
   (a) State what it means for a function $u : U \to \mathbb{R}$ to be weakly differentiable with respect to x_i $(1 \leq i \leq n)$.
   (b) Show that the function $u(x) = 1 - |x|$ is weakly differentiable on $U = (-1, 1)$.
4. State the definition of the Sobolev spaces $W^{k,p}(U)$ and the Sobolev norms $\|\cdot\|_{W^{k,p}(U)}$ for $k \in \mathbb{N}_0$ and $p \in [1, \infty]$.
5. Let $U \subset \mathbb{R}^n$ be open and bounded with C^1 boundary, and let $u \in W^{1,1}(U)$. Use the trace inequality to show that if there exists a sequence $\{u_j\}_{j=1}^{\infty} \subset C_c^{\infty}(U)$ satisfying $\lim_{j\to\infty} \|u_j - u\|_{W^{1,1}(U)} = 0$, then the trace of u on ∂U is 0.

6. Let $U \subset \mathbb{R}^n$ be open, and let $u \in L^2(U)$. Show that $u \in H^1(U)$ if and only if there exists a constant $C > 0$ such that for each $j = 1, 2, \ldots, n$,

$$\left| \int_U u \frac{\partial \phi}{\partial x_j} \, dx \right| \leq C \|\phi\|_{L^2(U)}, \quad \forall \phi \in C_c^\infty(U). \tag{1}$$

7. Let $U \subset \mathbb{R}^n$ be open and bounded, and let $f \in L^2(U)$. Write down a weak formulation of the problem

$$\begin{aligned} -\Delta u + u &= f, \text{ in } U, \\ u &= 0, \text{ on } \partial U, \end{aligned}$$

and show that it has a unique weak solution.

8. Let $U \subset \mathbb{R}^n$ be open and bounded with C^1 boundary. Show that if $n = 3$ and $f \in L^{6/5}(U)$, then the map

$$\begin{aligned} \ell : H^1(U) &\to \mathbb{R} \\ v &\mapsto \int_U f v \, dx \end{aligned}$$

is a bounded linear functional on $H^1(U)$. (Hint: Use a Sobolev inequality.)