Name: ______________________________ ID: ______________

1. Compute

$$\int \left(2\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2t\,\mathbf{k}\right)\,dt.$$

Solution: Integrating each component, we find that this integral is equal to

$$2\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + t^2\,\mathbf{k} + \mathbf{C},$$

where $\mathbf{C}$ is a vector constant.

2. Consider the helix

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}.$$

Find

- the unit tangent vector,
- the unit normal vector,
- the binormal vector, and
- the curvature

at $t = 0$.

Solution: We compute

$$\begin{aligned}
\mathbf{r}'(t) &= -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}, \\
\left|\mathbf{r}'(t)\right| &= \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{1+1} = \sqrt{2}, \\
\mathbf{T}(t) &= \frac{1}{\sqrt{2}}\left(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}\right), \\
\mathbf{T}(0) &= \frac{1}{\sqrt{2}}(\mathbf{j} + \mathbf{k}), \\
\mathbf{T}'(t) &= \frac{1}{\sqrt{2}}\left(-\cos t\,\mathbf{i} - \sin t\,\mathbf{j}\right), \\
\mathbf{T}'(0) &= -\frac{1}{\sqrt{2}}\mathbf{i}, \\
\left|\mathbf{T}'(0)\right| &= \frac{1}{\sqrt{2}}\sqrt{1^2 + 0^2 + 0^2} = \frac{1}{\sqrt{2}}, \\
\mathbf{N}(0) &= \frac{\mathbf{T}'(0)}{|\mathbf{T}'(0)|} = -\mathbf{i}, \\
\kappa(0) &= \frac{|\mathbf{T}'(0)|}{|\mathbf{r}'(0)|} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \frac{1}{2}, \\
\mathbf{B}(0) &= \mathbf{T}(0) \times \mathbf{N}(0) = \frac{1}{\sqrt{2}}(\mathbf{j} + \mathbf{k}) \times (-\mathbf{i}) \\
&= -\frac{1}{\sqrt{2}}(\mathbf{j} \times \mathbf{i} + \mathbf{k} \times \mathbf{i}) = -\frac{1}{\sqrt{2}}(-\mathbf{k} + \mathbf{j}) = \frac{1}{\sqrt{2}}(\mathbf{k} - \mathbf{j}).
\end{aligned}$$

Alternatively, we can find the curvature via

$$\begin{aligned}
\mathbf{r}'(0) &= \mathbf{j} + \mathbf{k}, \\
\mathbf{r}''(t) &= -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}, \\
\mathbf{r}''(0) &= -\mathbf{i}, \\
\mathbf{r}'(0) \times \mathbf{r}''(0) &= -\mathbf{j} \times \mathbf{i} - \mathbf{k} \times \mathbf{i} = \mathbf{k} - \mathbf{j}, \\
\left|\mathbf{r}'(0) \times \mathbf{r}''(0)\right| &= \sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{2}, \\
\kappa(0) &= \frac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}.
\end{aligned}$$