Name: ______________________________ ID: ________________

1. Find the exact length of the curve described by the polar equation

$$r = \theta^2, \qquad 0 \le \theta \le 2\pi.$$

u-substitution will help with computing the antiderivative.

Solution: We use the formula for the length of a polar curve:

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{\theta^2 + 4}\, \theta\, d\theta. \end{aligned}$$

At this point, we use the u-substitution $u = \theta^2 + 4$, so $du = 2\theta\, d\theta$. The bound $\theta = 0$ corresponds to $u = 4$, and the bound $\theta = 2\pi$ corresponds to $u = 4\pi^2 + 4$. We thus continue our computation:

$$\begin{aligned} L &= \int_4^{4\pi^2+4} \sqrt{u}\, \tfrac{1}{2}\, du \\ &= \tfrac{2}{3} u^{3/2}\, \tfrac{1}{2} \Big|_4^{4\pi^2+4} \\ &= \tfrac{1}{3}\left((4\pi^2 + 4)^{3/2} - 4^{3/2}\right) \\ &= \tfrac{1}{3}\left((4\pi^2 + 4)^{3/2} - 8\right) \\ &= \tfrac{8}{3}\left((\pi^2 + 1)^{3/2} - 1\right). \end{aligned}$$

The last two lines are both reasonably simplified answers.

2. (a) Describe the shape formed by the points (x, y, z) which satisfy both of the equations

$$x^2 + y^2 = 1 \qquad \text{and} \qquad z = 3.$$

Describe the shape in detail, including its size and location.

Solution: The equation $x^2 + y^2 = 1^2$ describes points (x, y, z) that are at a distance 1 from the z-axis. Along with the equation $z = 3$, we see that we have a horizontal circle with center $(0, 0, 3)$ and radius 1.

(b) Describe the surface described by the equation $x^2 + y^2 = 1$. Again, describe the shape in detail, including its size and location.

Solution: In this case, the value of z is not specified, so the surface contains all points (x, y, z) that are at a distance of 1 from the z-axis, namely a cylinder around the z-axis of radius 1.