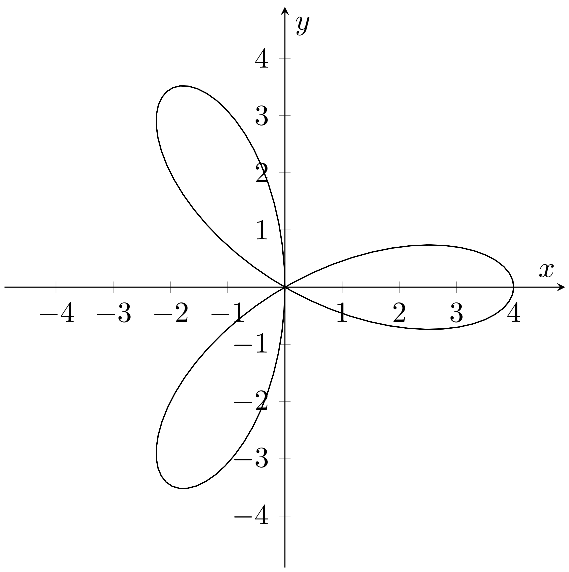
Name: ______________________ ID: ______________

1. Find the area of the region enclosed by one loop of the curve defined by the polar equation $r = 4\cos 3\theta$.



Solution: Looking at the angle the curve makes as it crosses the origin in the picture, we might guess that the loop on the right is traced out from $\theta = -\frac{\pi}{6}$ to $\theta = \frac{\pi}{6}$. To verify this guess, we note that $r = 0$ when $\cos 3\theta = 0$, which happens when $3\theta = \ldots, -\frac{5\pi}{2} - \frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$, so $\theta = \ldots, -\frac{5\pi}{6}, -\frac{\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6} \ldots$. So, indeed, at $\theta = -\frac{\pi}{6}$, the curve is at the origin. As θ increases, the curve moves away from the origin, and then it moves back to the origin at $\theta = \frac{\pi}{6}$, tracing out one loop in the process.

To compute the area, we can thus use our area formula

$$A = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2} r^2 \, d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}(4\cos 3\theta)^2 \, d\theta = 8\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2(3\theta) \, d\theta.$$

At this point, we might notice that $\cos^2\theta$ has period π, so $\cos^2(3\theta)$ has period $\frac{\pi}{3}$. Our interval of integration also has length $\frac{\pi}{3}$, so we are integrating $\cos^2(3\theta)$ over a full period, and so its average value is $\frac{1}{2}$. We can thus compute that

$$A = 8\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2} \, d\theta = 8 \cdot \frac{\pi}{3} \cdot \frac{1}{2} = \frac{4\pi}{3}.$$

Alternatively, we compute that

$$A = 8\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1 + \cos 6\theta}{2} \, d\theta = 4\left(\theta + \frac{1}{6}\sin 6\theta\right)\Bigg|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = 4\left(\frac{\pi}{6} + 0\right) - 4\left(-\frac{\pi}{6} + 0\right) = \frac{4\pi}{3}.$$

2. Sketch the graph of $9x^2 + 16y^2 = 144$ and locate the foci.

Solution: We divide by 144 to put the equation into a standard form. However, it's easier to compute that if we first express each constant as a square. We compute

$$3^2x^2 + 4^2y^2 = 12^2$$
$$\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1.$$

Thus, using our notation from class, $a = 4$ and $b = 3$, and we see that the major axis of the ellipse is along the x-axis. For an ellipse, c is smaller than a and has the formula $c^2 = a^2 - b^2 = 16 - 9 = 7$, so $c = \sqrt{7}$, somewhere between 2 and 3. Thus, our foci are at $(\pm\sqrt{7}, 0)$.

We can now sketch. We can trust our interpretation of the major and minor axis, or we can check from the equation that the maximum value of x is 4 and the maximum value of y is 3.