Math 243 Midterm 3

November 22, 2019

Name: ______________________________ ID: ______________

- Each page has a space at the top for the last 4 digits of your student ID. Make sure that you fill that out on at least one side of every sheet of paper.
- Show enough work that your solution would convince your peers that your answer is correct.
- The questions are ordered by topic, not by difficulty.
- Each question is worth the same number of points.
- You may not use any tools or resources other than writing implements. In particular, no calculators, phones, notes, and so forth.

1. Find the volume of the parallelepiped determined by the vectors $\mathbf{i}+\mathbf{j}$, $\mathbf{j}+\mathbf{k}$, and $\mathbf{i}+\mathbf{j}+\mathbf{k}$.

Solution: We can compute the volume of this parallelepiped using the scalar triple product of these vectors, which we can compute using the determinant

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1+0+1-1-0-0=1.$$

2. The line L_1 has equation

$$x = y = z.$$

The line L_2 has equation

$$x + 1 = \frac{y}{2} = \frac{z}{3}.$$

Show that the two lines are skew.

Solution: We first show that the two lines do not intersect. If a point is on both of the lines, then $x = y = z$, so substituting x for y and z in the equation of L_2, we find that

$$x + 1 = \frac{x}{2} = \frac{x}{3}.$$

The equation $\frac{x}{2} = \frac{x}{3}$ implies that $\frac{x}{2} - \frac{x}{3} = 0$, which simplifies to $\frac{x}{6} = 0$, so $x = 0$. On the other hand, plugging in $x = 0$ into the equation $x + 1 = \frac{x}{2}$ gives us $1 = 0$, a contradiction. Thus, there cannot be a point on both of these lines, so they do not intersect. We conclude that the lines are either parallel or skew.

In the notation from class, we can see that for L_1, $a = 1$, $b = 1$, and $c = 1$. Meanwhile, for L_2, $a = 1$, $b = 2$, and $c = 3$. The vectors $\langle 1, 1, 1\rangle$ and $\langle 1, 2, 3\rangle$ are not parallel to each other, so the two lines are not parallel.

We conclude that the lines are skew.

3. Consider the surface

$$3x^2 + y + 3z^2 = 0.$$

(a) Identify the horizontal traces of the surface in the planes $z = k$. If the answer depends on the value of k, be sure to specify which values of k give which answer.

Solution: Plugging in $z = k$, we find that

$$3x^2 + y + 3k^2 = 0,$$
$$y = -3x^2 - 3k^2.$$

This is the equation of a parabola for all values of k.

(b) Identify the vertical traces of the surface in the planes $x = k$. If the answer depends on the value of k, be sure to specify which values of k give which answer.

Solution: Plugging in $x = k$, we find that

$$3k^2 + y + 3z^2 = 0,$$
$$y = -3z^2 - 3k^2.$$

This is the equation of a parabola for all values of k.

(c) Identify the vertical traces of the surface in the planes $y = k$. If the answer depends on the value of k, be sure to specify which values of k give which answer.

Solution: Plugging in $y = k$, we find that

$$3x^2 + k + 3z^2 = 0,$$
$$3x^2 + 3z^2 = -k.$$

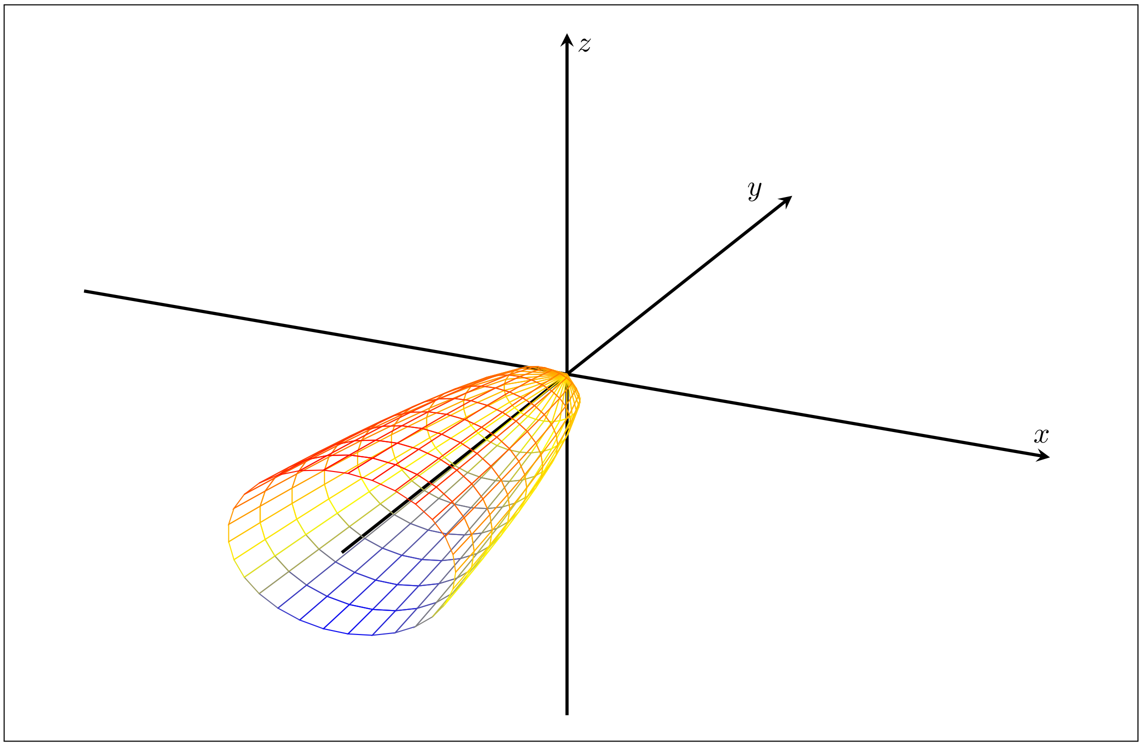
This is an equation of a circle as long as $-k > 0$, that is, $k < 0$.
If $k = 0$, then the equation $3x^2 + 3z^2 = 0$, whose only solution is the point $(x, z) = (0, 0)$.
If $k > 0$, then $3x^2 + 3z^2 < 0$, which is impossible, so the trace contains no points.

(d) Identify the surface.

Solution: A surface with parabolic and elliptic traces is an elliptic paraboloid.

(e) Sketch the surface. Your sketch does not need to be quantitatively correct, but it should show the correct type of surface in the correct orientation. If you feel like you need to, feel free to write a sentence to clarify the orientation.

Solution: Based on the traces with $y = k$, we see that the paraboloid opens in the negative y direction. Indeed, when $k > 0$, our traces are empty, when $k = 0$, our trace is a point, and when $k < 0$, our traces are circles.


z
y
x

4. Consider the circular cylinder $x^2+y^2=4$ and the parabolic cylinder $z=x^2$. These two surfaces intersect in a curve. Find parametric equations for this curve.

Solution: The projection of this curve onto the xy-plane is the circle $x^2+y^2=4$. This circle has radius 2, so one possible parametrization of this circle is $x=2\cos t$, $y=2\sin t$. For the curve in three dimensions, we have $z=x^2=4\cos^2 t$. One possible parametrization of the curve of intersection is thus

$$x = 2\cos t, \qquad y = 2\sin t, \qquad z = 4\cos^2 t.$$

5. Let $\mathbf{u}$ be a space curve that has $\mathbf{u}(2) = \langle 1, 2, -1\rangle$ and $\mathbf{u}'(2) = \langle 3, 0, 4\rangle$. Let $\mathbf{v}$ be the space curve defined by $\mathbf{v}(t) = \langle t, t^2, t^3\rangle$. Let $f(t) = \mathbf{u}(t) \cdot \mathbf{v}(t)$. Compute $f'(2)$.

Solution: The product rule tells us that

$$f'(2) = \mathbf{u}'(2) \cdot \mathbf{v}(2) + \mathbf{u}(2) \cdot \mathbf{v}'(2).$$

We compute that

$$\begin{aligned} \mathbf{v}(2) &= \langle 2, 4, 8\rangle, \\ \mathbf{v}'(t) &= \langle 1, 2t, 3t^2\rangle, \\ \mathbf{v}'(2) &= \langle 1, 4, 12\rangle. \end{aligned}$$

Thus, we compute that

$$\begin{aligned} f'(2) &= \langle 3, 0, 4\rangle \cdot \langle 2, 4, 8\rangle + \langle 1, 2, -1\rangle \cdot \langle 1, 4, 12\rangle \\ &= 6 + 0 + 32 + 1 + 8 - 12 \\ &= 35. \end{aligned}$$

6. Set up an integral for the length of the curve defined by

$$\mathbf{r}(t) = \langle \cos(\pi t), 2t, \sin(2\pi t) \rangle$$

from $(1, 0, 0)$ to $(1, 4, 0)$. You do not need to evaluate the integral.

Solution: We compute

$$\mathbf{r}'(t) = \langle -\pi \sin(\pi t), 2, 2\pi \cos(2\pi t) \rangle,$$
$$\left|\mathbf{r}'(t)\right| = \sqrt{\pi^2 \sin^2(\pi t) + 4 + 4\pi^2 \cos^2(2\pi t)}.$$

Now that we have the speed, we address the time bounds. The y component is the easiest one to deal with. When we are $(1, 0, 0)$, we have $y = 2t = 0$, so $t = 0$. Meanwhile, when we are at $(1, 4, 0)$, we have $y = 2t = 4$, so $t = 2$.

To find the distance traveled, we integrate the speed, obtaining a length of

$$L = \int_0^2 \sqrt{\pi^2 \sin^2(\pi t) + 4 + 4\pi^2 \cos^2(2\pi t)}\, dt.$$

There does not seem to be an easy way to simplify this further, though we could attempt to use double angle formulas if we needed to.