Math 243 Midterm 2

October 25, 2019

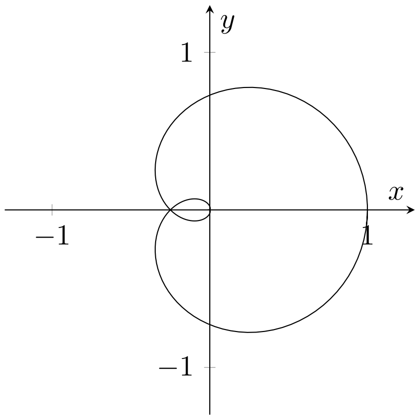
Name: ______________________________ ID: ________________

- Each page has a space at the top for the last 4 digits of your student ID. Make sure that you fill that out on at least one side of every sheet of paper.
- Show enough work that your solution would convince your peers that your answer is correct.
- The questions are ordered by topic, not by difficulty.
- Each question is worth the same number of points.
- You may not use any tools or resources other than writing implements. In particular, no calculators, phones, notes, and so forth.

1. Set up an integral for the length of the curve represented by the polar equation

$$r = \cos^4(\theta/4).$$

You do not need to evaluate the integral.



Solution: We use the formula

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

We compute that

$$\frac{dr}{d\theta} = 4\cos^3(\theta/4) \cdot (-\sin(\theta/4)) \cdot \tfrac{1}{4} = -\cos^3(\theta/4)\sin(\theta/4).$$

We also need to know the bounds of integration. The curve starts at $\theta = 0$, $r = 1$. When does it get back to where it started? We must find the next time θ is an even multiple of π and $r = 1$ or the next time θ is an odd multiple of π and $r = -1$. For this problem, the second case can't happen because $\cos^4(\theta/4) \geq 0$.

We check that at $\theta = 2\pi$, we have $r = \cos^4(\theta/4) = \cos^4(\pi/2) = 0$, so the curve is at the origin then, not at its start point. Indeed, from the diagram, we see that we've only traced half of the curve for $0 \leq \theta \leq 2\pi$. But at $\theta = 4\pi$, we compute that $r = \cos^4(\theta/4) = \cos^4(\pi) = (-1)^4 = 1$, so we are back to the start of the curve.

Thus, the integral for the arclength is

$$\begin{aligned} L &= \int_0^{4\pi} \sqrt{\cos^8(\tfrac{\theta}{4}) + \cos^6(\tfrac{\theta}{4})\sin^2(\tfrac{\theta}{4})}\, d\theta \\ &= \int_0^{4\pi} \sqrt{\cos^6(\tfrac{\theta}{4})\left(\cos^2(\tfrac{\theta}{4}) + \sin^2(\tfrac{\theta}{4})\right)}\, d\theta \\ &= \int_0^{4\pi} \sqrt{\cos^6(\tfrac{\theta}{4})}\, d\theta \\ &= \int_0^{4\pi} \left|\cos^3(\tfrac{\theta}{4})\right|\, d\theta \\ &= 2\int_0^{2\pi} \cos^3(\tfrac{\theta}{4})\, d\theta. \end{aligned}$$

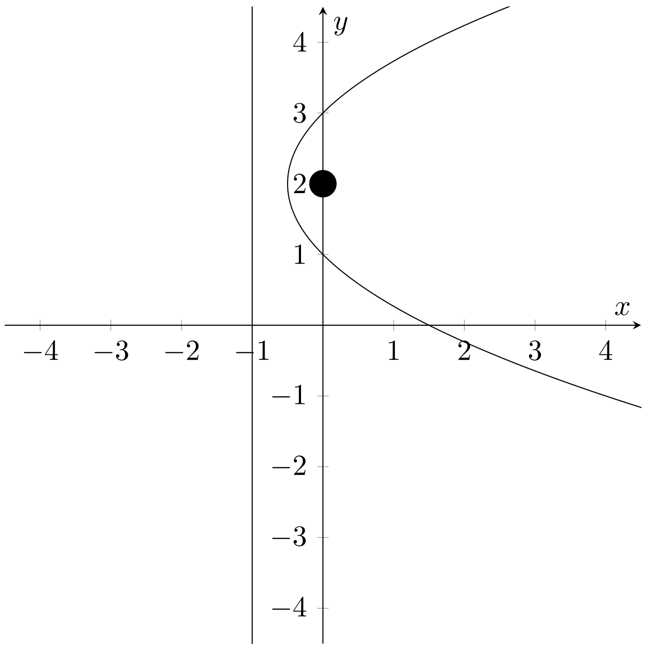
We could evaluate this integral with techniques from calculus II.

2. Find the vertex, focus, and directrix of the parabola

$$(y-2)^2 = 2x + 1.$$

Sketch the parabola, focus, and directrix.

The textbook contains the formula $x^2 = 4py$.



Solution: Factoring, we can rewrite the equation as

$$(y-2)^2 = 2(x + \tfrac{1}{2}).$$

Thus, we are working with the parabola $y^2 = 2x$ that has been shifted up by 2 and left by $\frac{1}{2}$. We conclude that the vertex of the parabola is at $(-\frac{1}{2}, 2)$.

By plotting a few more points, we can sketch the parabola. For example, when $x = 0$, then $(y-2)^2 = 2\frac{1}{2} = 1$, so $y - 2 = \pm 1$, and $y = 1$ or $y = 3$, so we can plot the points $(0, 1)$ and $(0, 3)$.

The parabola opens to the right, so we need to swap x and y in the textbook formula to obtain $y^2 = 4px$. Comparing this expression to $y^2 = 2x$, we see that $p = \frac{1}{2}$. The quantity p represents the distance from the vertex to the focus, which is also the distance from the focus to the directrix.

Our vertex is at $(-\frac{1}{2}, 2)$, so the focus is $\frac{1}{2}$ to the right at $(0, 2)$. Meanwhile, the directrix is $\frac{1}{2}$ to the left of the vertex, so it is the vertical line $x = -1$.

3. Find the vertices and foci of the curve described by the equation

$$\frac{x^2}{2} + \frac{y^2}{4} = 1.$$

Sketch the curve, label the vertices, and draw and label the foci.

Solution: We can rewrite this equation in the standard form for an ellipse:

$$\frac{x^2}{(\sqrt{2})^2} + \frac{y^2}{2^2} = 1.$$

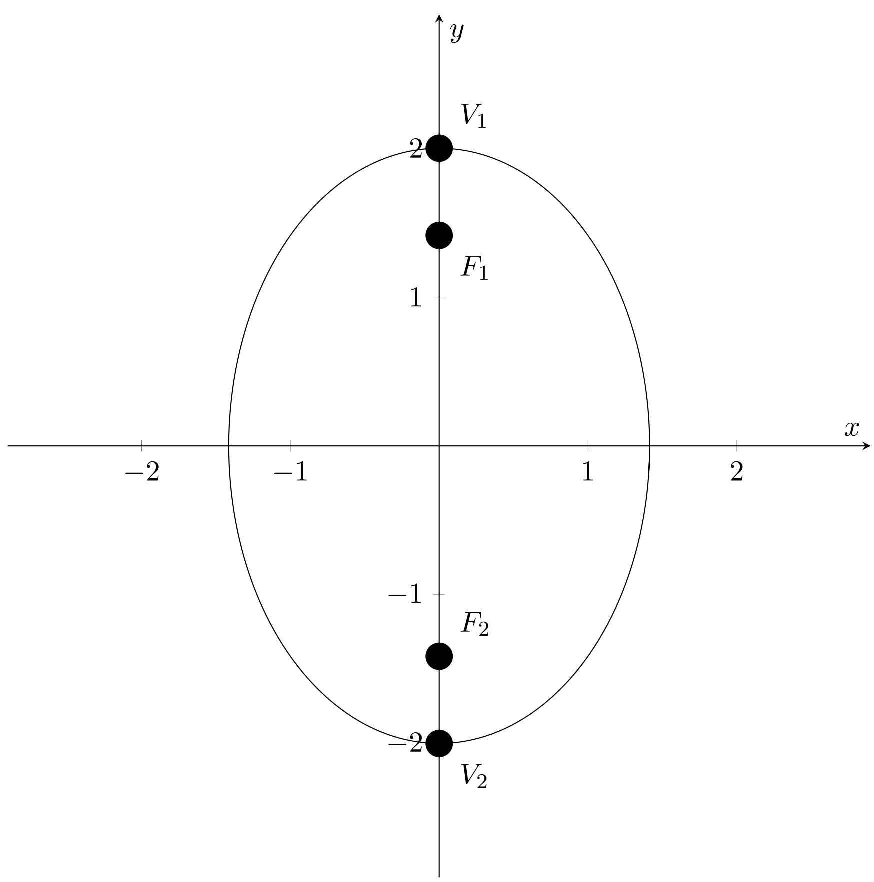
Since $2 > \sqrt{2}$, we see that $a = 2$ and $b = \sqrt{2}$. Since $a = 2$ is associated to y, we have a vertical ellipse, with vertices at $(0, \pm 2)$. We can also check from the equation that the maximum possible value of y is 2. For our sketch, we will likewise need that x ranges from $-\sqrt{2}$ to $\sqrt{2}$

To find the foci, we compute that

$$c = \sqrt{a^2 - b^2} = \sqrt{4 - 2} = \sqrt{2}.$$

Because we have a vertical ellipse, the foci are on the y-axis at $(0, \pm\sqrt{2})$.

We now have all the information needed for our sketch.



4. Find the equation of the sphere with center $(2, -3, 6)$ that touches the xy-plane.

Solution: The point $(2, -3, 6)$ is six units above the xy-plane, so a sphere with this center that just touches the xy-plane has radius 6. Knowing the center and radius of the sphere let us write its equation using the distance formula:

$$(x-2)^2 + (y+3)^2 + (z-6)^2 = 6^2.$$

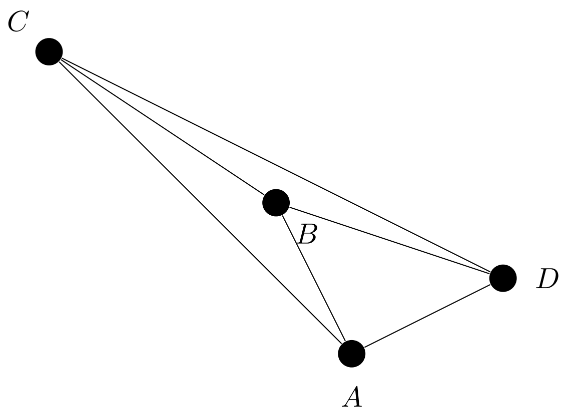
5. Describe in words the region of three-dimensional space represented by the equation

$$x^2 + y^2 + z^2 \leq 4.$$

Be specific.

Solution: The quantity $x^2 + y^2 + z^2$ represents the distance squared between $P(x, y, z)$ and the origin. Thus, for all points with $x^2 + y^2 + z^2 \leq 2^2$, the distance to the origin is less than or equal to 2. These points form a solid ball of radius two centered at the origin.

6. Consider four points below:



Write each combination of vectors as a single vector.

(a) $\overrightarrow{AB} + \overrightarrow{BC}$.

Solution: Moving from A to B and then from B to C is the same as moving from A to C, which is represented by the vector $\overrightarrow{AC}$.

(b) $\overrightarrow{CD} + \overrightarrow{DB}$.

Solution: Moving from C to D and the moving from D to B is the same as moving from C to B, which is represented by the vector $\overrightarrow{CB}$.

(c) $\overrightarrow{DB} - \overrightarrow{AB}$.

Solution: Moving from A to B in reverse is the same as moving from B to A. In other words, $-\overrightarrow{AB} = \overrightarrow{BA}$. Moving from D to B and then moving from B to A is the same as moving from D to A, which is represented by the vector $\overrightarrow{DA}$.

(d) $\overrightarrow{DC} + \overrightarrow{CA} + \overrightarrow{AB}$.

Solution: Moving from D to C, then C to A, and then A to B is the same as moving from D to B, which is represented by the vector $\overrightarrow{DB}$.

7. Find the work done by a force $\mathbf{F} = 8\mathbf{i} - 6\mathbf{j} + 9\mathbf{k}$ acting on an object that moves from the point $(0, 10, 8)$ to the point $(6, 12, 18)$ along a straight line. The distance is measured in meters and the force is measured in newtons.

Solution: We recall that work is the dot product of the force and the displacement vectors. The displacement vector is

$$\mathbf{D} = (6-0)\mathbf{i} + (12-10)\mathbf{j} + (18-8)\mathbf{k} = 6\mathbf{i} + 2\mathbf{j} + 10\mathbf{k}.$$

Thus,

$$W = \mathbf{F} \cdot \mathbf{D} = (8)(6) + (-6)(2) + (9)(10) = 48 - 12 + 90 = 126,$$

in joules.