Math 243 Midterm 1

September 27, 2019

Name: ______________________________ ID: ________________

- Each page has a space at the top for the last 4 digits of your student ID. Make sure that you fill that out on at least one side of every sheet of paper.
- Show enough work that your solution would convince your peers that your answer is correct.
- The questions are ordered by topic, not by difficulty.
- Each question is worth the same number of points.
- You may not use any tools or resources other than writing implements. In particular, no calculators, phones, notes, and so forth.

1. Set up the integral that represents the length of the curve described by the equation

$$y = x - \ln x, \qquad 1 \le x \le 4.$$

You do not need to evaluate the integral.

Solution: We use the formula for arclenth.

$$\begin{aligned} L &= \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_1^4 \sqrt{1 + \left(1 - \frac{1}{x}\right)^2}\, dx \\ &= \int_1^4 \sqrt{1 + 1 - \frac{2}{x} + \frac{1}{x^2}}\, dx \\ &= \int_1^4 \sqrt{2 - \frac{2}{x} + \frac{1}{x^2}}\, dx. \end{aligned}$$

There doesn't seem to be an easy way to simplify this further, so we heed the advice to not evaluate the integral and stop here.

2. Consider the curve described by the parametric equations

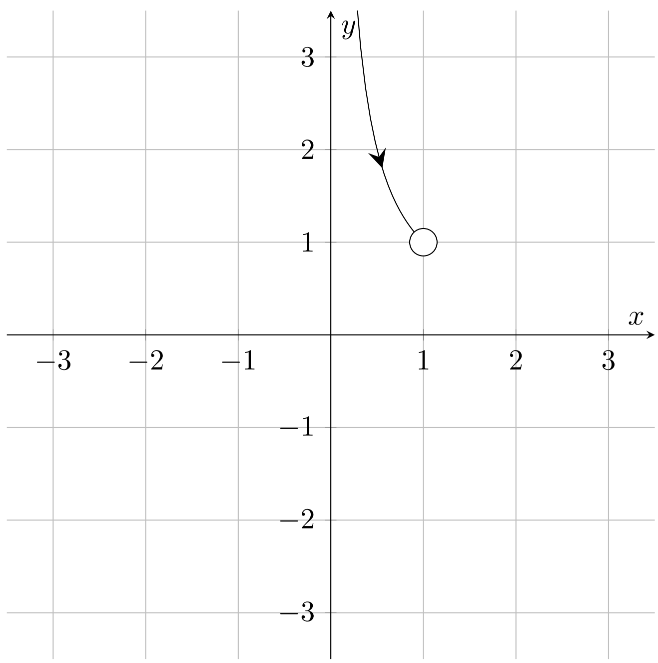
$$x = \sin t, \qquad y = \csc t, \qquad 0 < t < \tfrac{\pi}{2}.$$

(a) Eliminate the parameter to find a Cartesian equation of the curve.

Solution: We see that $y = \csc t = \frac{1}{\sin t} = \frac{1}{x}$, which is an expression that does not involve t, so we're done. Our Cartesian equation for the curve is

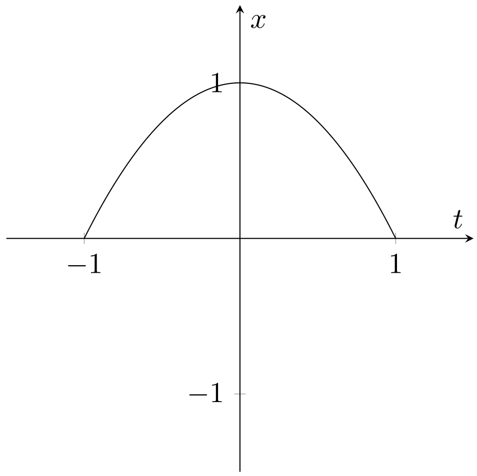
$$y = \frac{1}{x}.$$

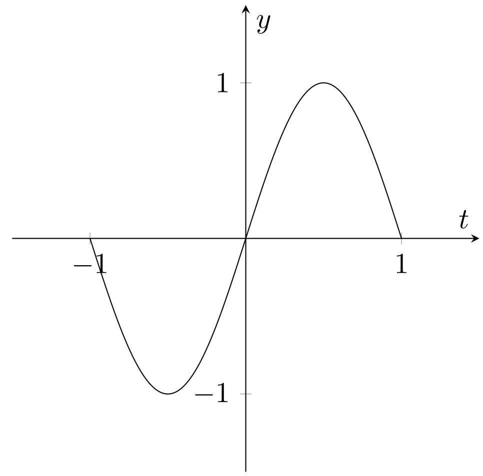
(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

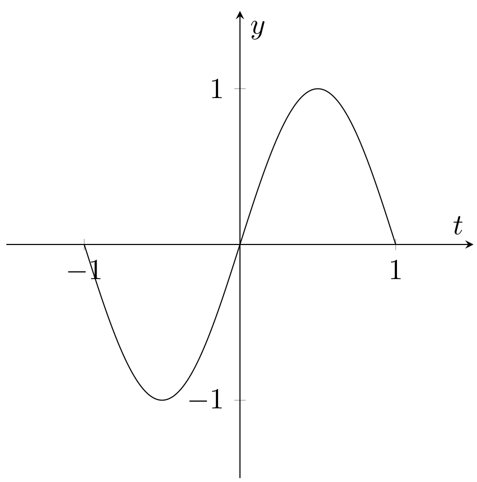


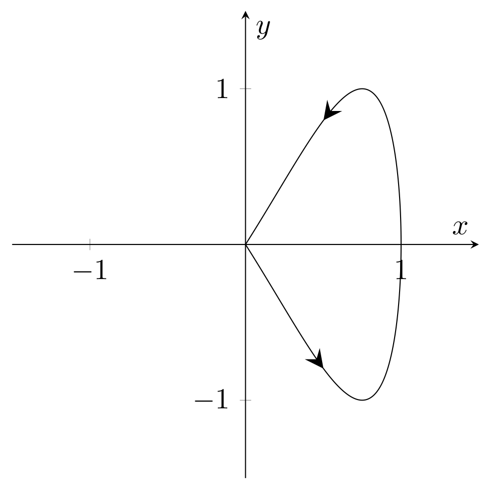
Solution: The tricky part of this problem is that we don't get the entire curve $y = \frac{1}{x}$. We see that, as t ranges from 0 to $\frac{\pi}{2}$, $\sin t$ ranges from 0 to 1, so we only plot the curve from $0 < x < 1$. This reasoning also tells us that we should place our arrow pointing rightwards.

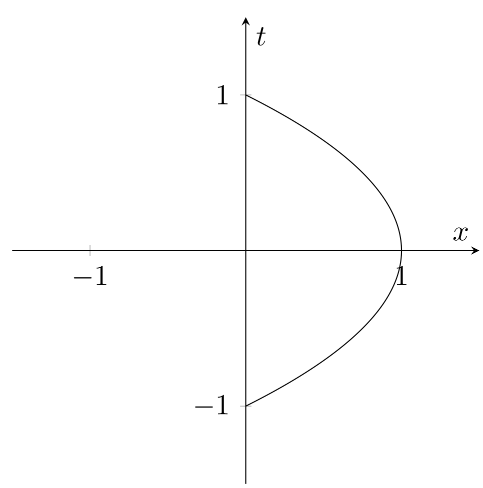
3. Use the provided graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve defined by these equations. Indicate with arrows the direction in which the curve is traced as t increases.







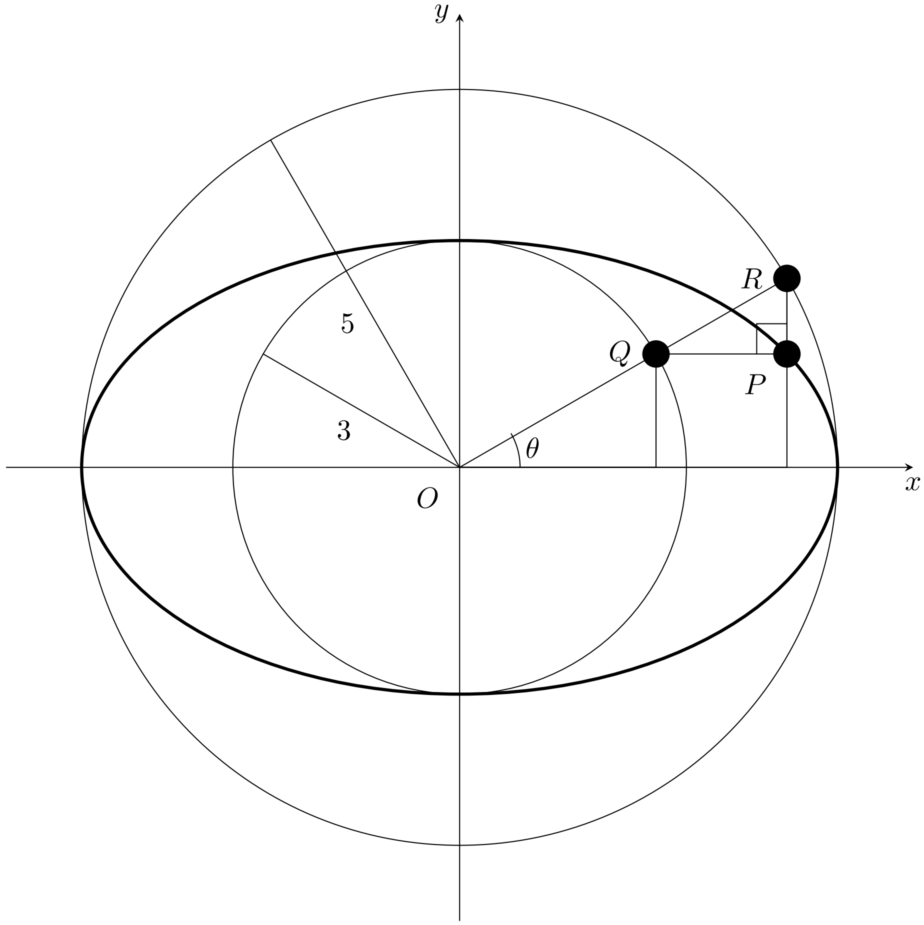




Solution: Following the method from class, we redraw the given plots to line up the y-axis of the ty plot with the y-axis of the xy plot, and line up the x-axis of the tx plot with the x-axis of the xy plot.

From these two plots, we see that, at $t = -1$, the curve is at $x = 0$ and $y = 0$, and from there it moves down and to the right. Around $t = -\frac{1}{2}$, it switches to moving up and to the right. At $t = 0$, the curve is at $(x, y) = (1, 0)$, and switches from moving up and to the right to moving up and to the left. At around $t = \frac{1}{2}$, it switches to moving down and to the left, and it continues moving down and to the left until it arrives at $(x, y) = (0, 0)$ at $t = 1$.

4. Find parametric equations that describe the curve traced out by the point P as θ varies. Then add a sketch of the curve to the diagram.



Solution: I've added points Q and R to the diagram, and added vertical lines joining them to the x-axis. We see that OQ has length 3 and is the hypotenuse of a right triangle with angle θ, which tells us that Q has coordinates $(3\cos\theta, 3\sin\theta)$. Meanwhile, OR has length 5 and is also the hypotenuse of a right triangle with angle θ, which tells us that R has coordinates $(5\cos\theta, 5\sin\theta)$. We see that P has the same x-coordinate as R and the same y-coordinate as Q, so P has coordinates $(5\cos\theta, 3\sin\theta)$. Thus, the parametric equations describing the curve traced out by P are

$$x = 5\cos\theta, \qquad y = 3\sin\theta.$$

We recognize these as the equations of a unit circle that has been stretched out by 5 in the x direction and by 3 in the y direction, namely an ellipse as shown.

If we like, we can verify our work by picking another angle θ and finding the location P, checking that it lies on the ellipse.

5. Consider the parametric curve defined by the equations

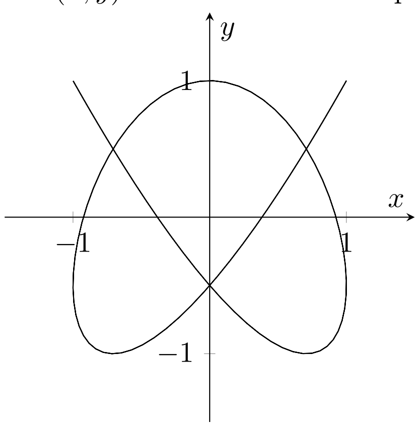
$$x = \cos 3\theta, \qquad y = \cos 4\theta.$$

(a) Compute $\frac{dy}{dx}$ in terms of θ.

Solution: Using our formula for the slope of a parametric curve, we compute that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-4\sin 4\theta}{-3\sin 3\theta} = \frac{4\sin 4\theta}{3\sin 3\theta}.$$

(b) The curve is plotted below, and you can see that there are three points where the tangent to the curve is horizontal and two points where the tangent to the curve is vertical. List the (x, y) coordinates of these points.



Solution: We expect horizontal tangents when the numerator of the above fraction is zero, which happens when $\sin 4\theta = 0$, which happens when $4\theta = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots$, which happens when

$$\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \ldots.$$

Meanwhile, we expect vertical tangents when the denominator of the above fraction is zero, which happens when $\sin 3\theta = 0$, which happens when $3\theta = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots$, which happens when

$$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \ldots.$$

In general, we should consider negative values of θ as well. Can you see why that wouldn't make a difference for this problem? Hint: Cosine is even.

When $\theta = 0$, we have a problem, because both the numerator and the denominator are zero, so we could have a horizontal tangent or a vertical tangent or neither. On an exam, I'd skip this value for now, but if we were to analyze it, we'd see that the location on the curve corresponding to $\theta = 0$ is $(x, y) = (\cos 0, \cos 0) = (1, 1)$, and we see that the curve is diagonal at $(1, 1)$. It's not one of the points we're looking for. If we had tried $\theta = \pi$, we'd end up with a similar situation, except the point we'd be looking at would be $(x, y) = (\cos 3\pi, \cos 4\pi) = (-1, 1)$, also not one of the points we're looking for.

On the other hand, the values $\theta = \frac{\pi}{4}$, $\theta = \frac{\pi}{2}$, and $\theta = \frac{3\pi}{4}$ look promising for finding our three points of horizontal tangency, and the values $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$ look promising for our three points of vertical tangency. Hopefully when we compute the points we won't get any repeats. If so, we'll be done.

We plug these values of θ into our parametric equations:

θ	$x = \cos 3\theta$	$y = \cos 4\theta$
$\frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$	-1
$\frac{\pi}{2}$	0	1
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	-1
$\frac{\pi}{3}$	-1	$-\frac{1}{2}$
$\frac{2\pi}{3}$	1	$-\frac{1}{2}$

We notice we didn't get any repeats, so we've found all five points, and we put them in the blanks. We check with the picture to make sure that the coordinates we found look about right. Do the (x, y) pairs we found seem to correspond to the locations of the points of horizontal and vertical tangency in the picture?

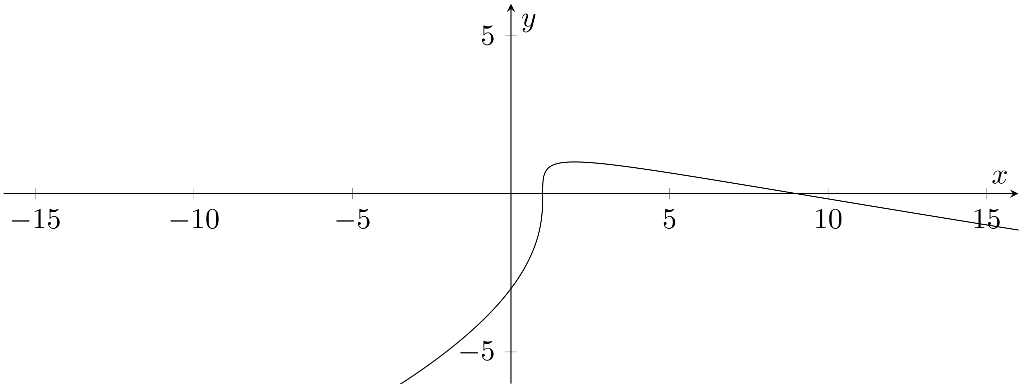
Points with horizontal tangents: $\left(\pm\frac{\sqrt{2}}{2}, -1\right), (0, 1)$

Points with vertical tangents: $\left(\pm 1, -\frac{1}{2}\right)$

6. Find the area enclosed by the x-axis and the curve defined by the parametric equations

$$x = t^3 + 1, \qquad y = 2t - t^2.$$

The curve is plotted below for reference.



Solution: Our formula for area under the curve tells us that

$$A = \int y\,dx = \int y\frac{dx}{dt}\,dt = \int \left(2t - t^2\right)\left(3t^2\right)\,dt = \int \left(6t^3 - 3t^4\right)\,dt.$$

We're missing bounds, though. We need to integrate between the two points where the curve crosses the x-axis. It crosses the x-axis when $y = 0$, so to compute the points of intersection we set $0 = 2t - t^2$. Factoring, we find that $t(2-t) = 0$, so $t = 0$ or $t = 2$.

If you're worried about whether we'll get the negative area, notice that $\frac{dx}{dt}$ is always positive or zero, and y is positive or zero in the region we're considering. We thus compute

$$A = \int_0^2 \left(6t^3 - 3t^4\right)\,dt = \left(\tfrac{3}{2}t^4 - \tfrac{3}{5}t^5\right)\Big|_{t=0}^{2} = 24 - \tfrac{96}{5} = \tfrac{24}{5} = 4.8.$$

We can roughly check our work against the picture. The region whose area we're computing looks roughly like a triangle with base 8-ish and height 1-ish, which has an area of 4, and 4.8 is pretty close to 4. If we had gotten an answer like 50 or 0.3, we would be worried.

7. Set up an integral that represents the length of the curve defined by the parametric equations

$$x = t + e^{-t}, \qquad y = t - e^{-t}, \qquad 0 \leq t \leq 2.$$

You do not need to evaluate the integral.

Solution: We use the formula for the arclength of a parametric curve.

$$\begin{aligned} L &= \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^2 \sqrt{(1-e^{-t})^2 + (1+e^{-t})^2}\, dt \\ &= \int_0^2 \sqrt{1 - 2e^{-t} + e^{-2t} + 1 + 2e^{-t} + e^{-2t}}\, dt \\ &= \int_0^2 \sqrt{2 + 2e^{-2t}}\, dt. \end{aligned}$$

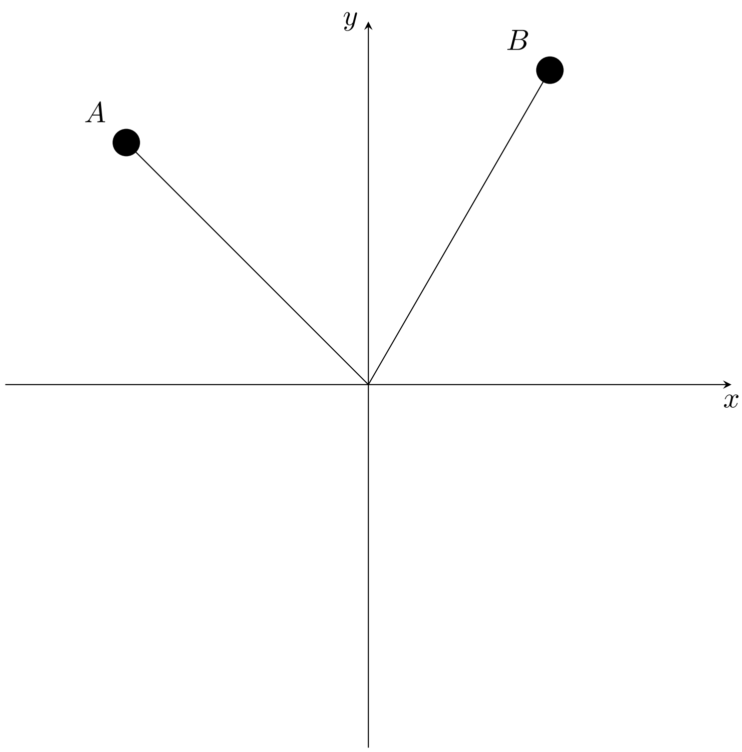
It doesn't seem that we can simplify this further, so we heed the advice to not evaluate the integral and stop here.

Fun fact: the value of this integral is $3.1415\ldots$. However, it is not equal to π, despite being quite close.

8. Consider the point A with Cartesian coordinates $(-4, 4)$, and the point B with Cartesian coordinates $(3, 3\sqrt{3})$.

   For each point, find *two* polar coordinates of the point.

Solution: We make rough sketches of these two points.



The key here is to draw the points in the correct quadrant and to make sure that $3\sqrt{3}$ is longer than 3.

We see from the picture that point A makes a $45°$ angle with the negative x-axis, so it lies at a θ value of $\frac{3\pi}{4}$. Its distance from the origin is $\sqrt{4^2 + 4^2} = 4\sqrt{1^2 + 1^2} = 4\sqrt{2}$. We thus obtain the polar coordinates $(r, \theta) = \left(4\sqrt{2}, \frac{3\pi}{4}\right)$.

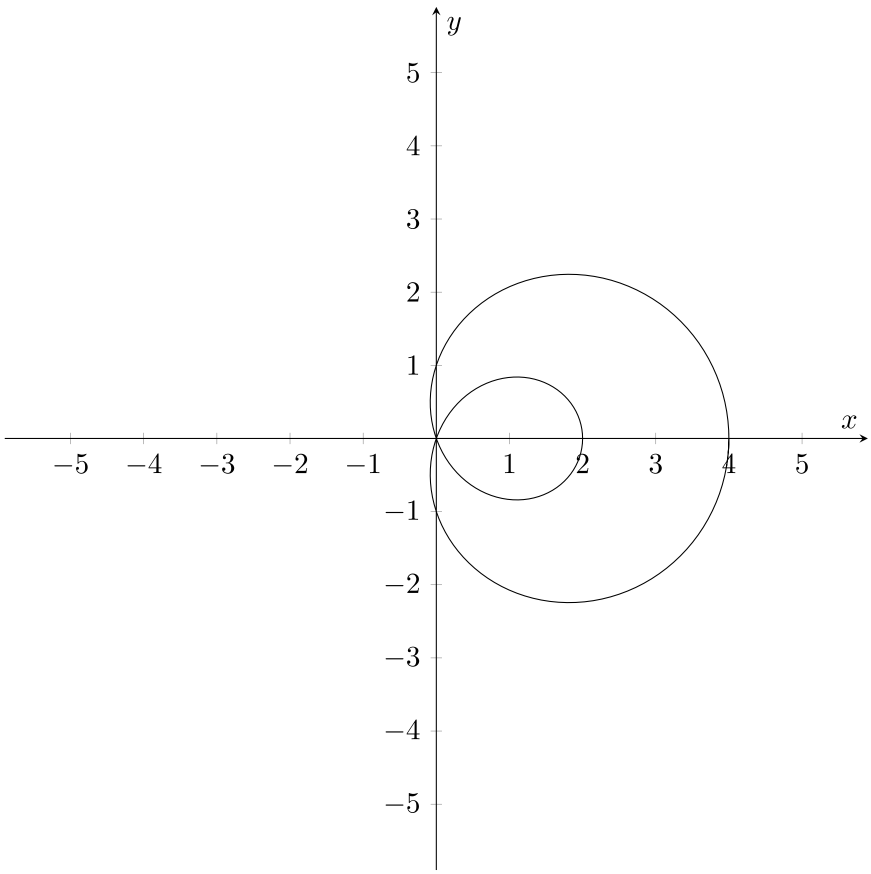
Meanwhile, B makes an angle with the positive x-axis that is bigger than $45°$, so we might guess that the angle is $60°$. We can confirm our guess using our knowledge of $30°$–$60°$–$90°$ triangles. This knowledge also tells us that the hypotenuse is twice as long as the short side, making its length 6. Alternatively, we use the distance formula, which tells us that the distance from B to the origin is $\sqrt{3^2 + (3\sqrt{3})^2} = 3\sqrt{1^2 + (\sqrt{3})^2} = 3\sqrt{1+3} = 6$. We thus obtain polar coordinates $(r, \theta) = \left(6, \frac{\pi}{3}\right)$.

The question asks for two polar coordinates for each point. To get a second polar coordinate for the point, we can add or subtract 2π to the angle, or we can switch the sign of the radius and add or subtract π to the angle. I'll do the latter because I find adding or subtracting π easier than adding or subtracting 2π. The result is $(r, \theta) = \left(-4\sqrt{2}, -\frac{\pi}{4}\right)$ for A, and $(r, \theta) = (-6, -\frac{2\pi}{3})$ for B.

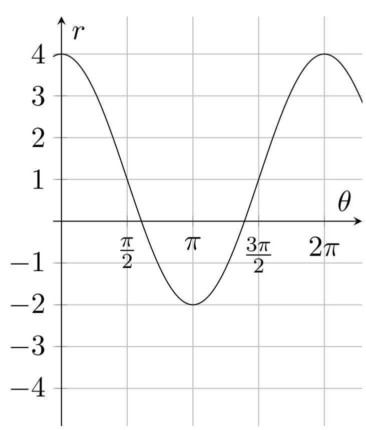
9. Sketch the curve described by the polar equation

$$r = 1 + 3\cos\theta.$$

You may find it helpful to first make a separate graph of r as a function of θ in Cartesian coordinates.



Solution: As suggested, we sketch a Cartesian plot of r versus θ in Cartesian coordinates. It's a cosine graph stretched vertically by a factor of 3 and shifted upwards by 1 unit. Its maximum value should thus be $1+3(1) = 4$, its minimum value should be $1+3(-1) = -2$, and its average value should be 1.



When $\theta = 0$, we have $r = 4$, so we start our plot at 4 on the x-axis. We see that the radius decreases, and at $\theta = \frac{\pi}{2}$ we have $r = 1$, so we make sure to cross the y-axis at $r = 1$. Shortly thereafter, r decreases to zero, so our curve crosses through the origin.

At this point, r is now negative, so instead of plotting in the second quadrant, we have crossed over to the fourth quadrant. The value of r continues to decrease, which means that $|r|$ is increasing, so our curve moves further from the origin. This continues until we get to $\theta = \pi$, where $r = -2$. This corresponds to the point $x = 2$ on the x-axis.

We could continue plotting as before, or we could notice that cosine is even so $1+3\cos(-\theta) = 1+3\cos(\theta)$. As discussed in class, we can conclude that our plot should be symmetric with respect to reflection across the x-axis, so we can reflect what we've drawn across the x-axis to finish the plot, without having to painstakingly consider values of θ from π to 2π. Meanwhile, values of θ beyond 2π don't provide any new points on the curve because $1 + 3\cos\theta$ is periodic with period 2π.

10. Find the slope of the tangent line to the curve described by the polar equation $r = 2\cos\theta$ at the point specified by $\theta = \pi/3$.

Solution: We use the formulas

$$x = r\cos\theta, \qquad y = r\sin\theta$$

to compute

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{(-2\sin\theta)\sin\theta + (2\cos\theta)\cos\theta}{(-2\sin\theta)\cos\theta - 2\cos\theta\sin\theta} = -\frac{\cos^2\theta - \sin^2\theta}{2\sin\theta\cos\theta}.$$

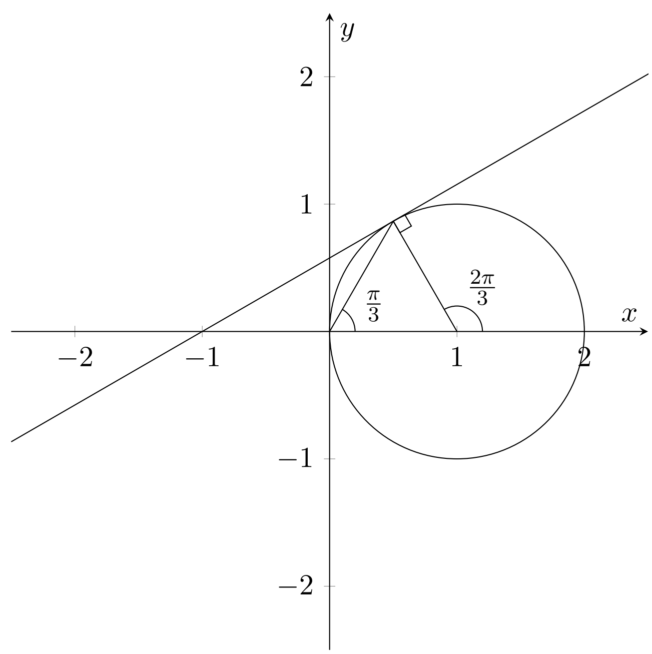
We could plug in $\theta = \frac{\pi}{3}$ directly into this formula, but it's faster to use the double angle formulas first.

$$\frac{dy}{dx} = -\frac{\cos 2\theta}{\sin 2\theta} = -\cot 2\theta.$$

Plugging in $\theta = \frac{\pi}{3}$, we see that the solve of the tangent line at $\theta = \frac{\pi}{3}$ is

$$-\cot\frac{2\pi}{3} = -\left(-\frac{1}{\sqrt{3}}\right) = \frac{\sqrt{3}}{3}.$$

Alternatively, or as a way of checking our work, we could recognize this as the polar equation of a circle with diameter 2 centered on the x-axis at $x = 1$, and compute the slope of the tangent line using geometry.



We see that the x-axis, the y-axis, and the tangent line form a 30°–60°–90° triangle. In such a triangle, the long leg is $\sqrt{3}$ times longer than the short leg. The short leg is the "rise" in this picture and the long leg is the "run," so the slope is $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

Alternatively, the radius of the circle in the picture is at an angle of $\frac{2\pi}{3}$, so its slope is $\tan\frac{2\pi}{3} = -\sqrt{3}$. Using the formula for the slope of the perpendicular line, we find that our desired slope is $-\frac{1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}$.