Math 243 Final

December 20, 2019

Name: ______________________________ ID: ______________

- Each page has a space at the top for the last 4 digits of your student ID. Make sure that you fill that out on at least one side of every sheet of paper.
- Show enough work that your solution would convince a skeptical peer that your answer is correct.
- The questions are ordered by topic, not by difficulty.
- Each question is worth the same number of points.
- You may not use any tools or resources other than writing implements and a 3 inch by 5 inch note card. In particular, no calculators, phones, additional notes, and so forth.

1. Set up an integral for the length of the curve described by the equations

$$x = t - e^t, \qquad y = t + e^t$$

from $t = -6$ to $t = 6$. You do not need to evaluate the integral.

Solution: In terms of vectors, we can easily compute the position, velocity, and speed to be

$$\begin{aligned}\mathbf{r}(t) &= \langle t - e^t, t + e^t \rangle \\ \mathbf{v}(t) &= \langle 1 - e^t, 1 + e^t \rangle \\ v(t) &= \sqrt{(1-e^t)^2 + (1+e^t)^2} = \sqrt{2 + 2e^{2t}}.\end{aligned}$$

Integrating speed with respect to time, we find that the length travelled is

$$\int_{-6}^{6} \sqrt{2 + 2e^{2t}}\, dt$$

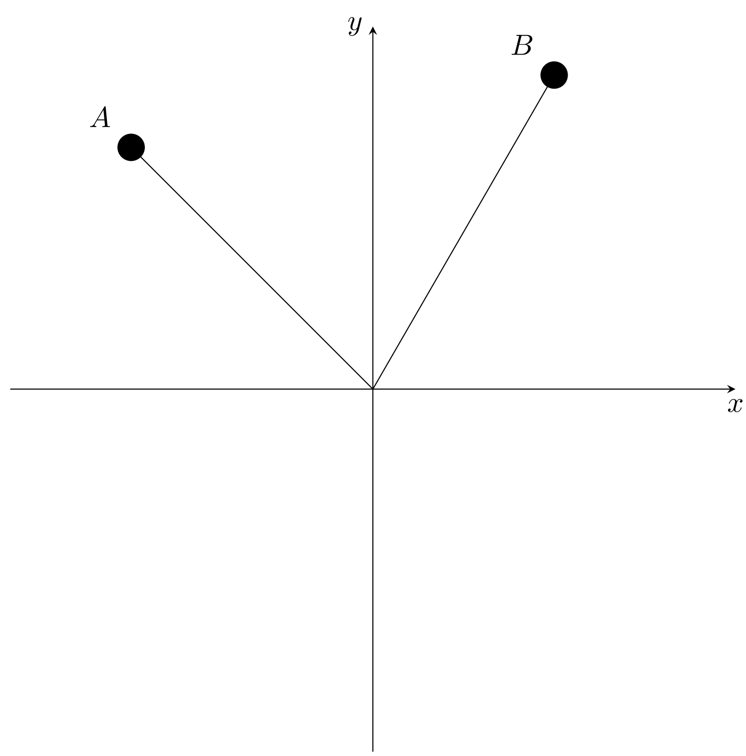
Alternatively, we can do the same computations and obtain the same answer using the formula for arclength

$$L = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

2. Consider the point A with Cartesian coordinates $(-4, 4)$, and the point B with Cartesian coordinates $(3, 3\sqrt{3})$.

   For each point, find *two* polar coordinates of the point.

Solution: We make rough sketches of these two points.



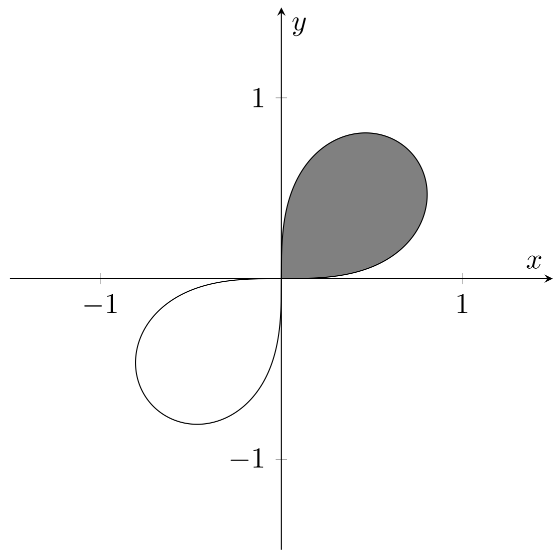
The key here is to draw the points in the correct quadrant and to make sure that $3\sqrt{3}$ is longer than 3.

We see from the picture that point A makes a 45° angle with the negative x-axis, so it lies at a θ value of $\frac{3\pi}{4}$. Its distance from the origin is $\sqrt{4^2+4^2} = 4\sqrt{1^2+1^2} = 4\sqrt{2}$. We thus obtain the polar coordinates $(r,\theta) = \left(4\sqrt{2}, \frac{3\pi}{4}\right)$.

Meanwhile, B makes an angle with the positive x-axis that is bigger than 45°, so we might guess that the angle is 60°. We can confirm our guess using our knowledge of 30°–60°–90° triangles. This knowledge also tells us that the hypotenuse is twice as long as the short side, making its length 6. Alternatively, we use the distance formula, which tells us that the distance from B to the origin is $\sqrt{3^2+(3\sqrt{3})^2} = 3\sqrt{1^2+(\sqrt{3})^2} = 3\sqrt{1+3} = 6$. We thus obtain polar coordinates $(r,\theta) = \left(6, \frac{\pi}{3}\right)$.

The question asks for two polar coordinates for each point. To get a second polar coordinate for the point, we can add or subtract 2π to the angle, or we can switch the sign of the radius and add or subtract π to the angle. I'll do the latter because I find adding or subtracting π easier than adding or subtracting 2π. The result is $(r,\theta) = \left(-4\sqrt{2}, -\frac{\pi}{4}\right)$ for A, and $(r,\theta) = (-6, -\frac{2\pi}{3})$ for B.

3. Find the area of the shaded region.



$r^2 = \sin 2\theta$

Solution: The formula for area in polar coordinates is $A = \frac{1}{2}\int_a^b r^2\, d\theta$, based on the area of a sector with angle $d\theta$ of a circle of radius r.

To find the bounds, we can see from the picture that the curve defining the shaded region is traced out from $\theta = 0$ to $\theta = \frac{\pi}{2}$. Alternatively, we see that the curve tracing out the shaded region starts and ends at the origin. We note that $r = 0$ when $\theta = 0$. Thereafter, $\sin 2\theta$ becomes positive, so r is not zero. The next time $r = 0$ happens the next time $\sin 2\theta = 0$, which happens when $2\theta = \pi$, or $\theta = \frac{\pi}{2}$.

Plugging in the bounds and $r^2 = \sin 2\theta$ into the formula, we compute that

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta\, d\theta = -\frac{1}{4}\cos 2\theta\Big|_0^{\frac{\pi}{2}} = -\frac{1}{4}(\cos\pi - \cos 0) = -\frac{1}{4}(-1-1) = \frac{1}{2}.$$

4. Let $\mathbf{a} = \langle 4, 1, \frac{1}{4} \rangle$ and $\mathbf{b} = \langle 6, -3, -8 \rangle$. Compute $\mathbf{a} \cdot \mathbf{b}$.

Solution: We compute

$$\mathbf{a} \cdot \mathbf{b} = (4)(6) + (1)(-3) + (\tfrac{1}{4})(-8) = 24 - 3 - 2 = 19.$$

5. Consider the space curve
$$\mathbf{r}(t) = \sin^2 t\,\mathbf{i} + \cos^2 t\,\mathbf{j} + \tan^2 t\,\mathbf{k}.$$
Find the unit tangent tangent vector at $t = \frac{\pi}{4}$.

Solution: Using the chain rule and the quotient rule, we compute that

$$\begin{aligned}
&\frac{d}{dt}\sin^2 t = 2\sin t\cos t,\\
&\frac{d}{dt}\cos^2 t = -2\cos t\sin t,\\
&\frac{d}{dt}\tan^2 t = 2\tan t\frac{d}{dt}\frac{\sin t}{\cos t} = 2\tan t\frac{\cos t\cos t - \sin t(-\sin t)}{\cos^2 t} = 2\tan t\frac{1}{\cos^2 t}
\end{aligned}$$

Thus, we can compute

$$\begin{aligned}
\mathbf{r}'(t) &= 2\sin t\cos t\,\mathbf{i} - 2\cos t\sin t\,\mathbf{j} + 2\frac{\tan t}{\cos^2 t}\,\mathbf{k},\\
\mathbf{r}'\left(\tfrac{\pi}{4}\right) &= 2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\mathbf{i} - 2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\,\mathbf{j} + 2\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2}\,\mathbf{k} = \mathbf{i} - \mathbf{j} + 4\mathbf{k},\\
\left|\mathbf{r}'\left(\tfrac{\pi}{4}\right)\right| &= \sqrt{1+1+16} = 3\sqrt{2},\\
\mathbf{T}\left(\tfrac{\pi}{4}\right) &= \frac{\mathbf{r}'\left(\frac{\pi}{4}\right)}{\left|\mathbf{r}'\left(\frac{\pi}{4}\right)\right|} = \frac{1}{3\sqrt{2}}(\mathbf{i} - \mathbf{j} + 4\mathbf{k}).
\end{aligned}$$

6. Find the curvature of the graph of the function $y = xe^x$.

Solution: A point on the graph of this function is of the form $(x, y) = (x, xe^x)$. Thus, we compute

$$\begin{aligned} \mathbf{r} &= \langle x, xe^x \rangle, \\ \mathbf{r}' &= \langle 1, e^x + xe^x \rangle, \\ \mathbf{r}'' &= \langle 0, e^x + e^x + xe^x \rangle = \langle 0, 2e^x + xe^x \rangle. \end{aligned}$$

From here, we can one of our formulas for curvature to compute

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{|(1)(2e^x + xe^x) - (e^x + xe^x)(0)|}{\left(1 + (e^x + xe^x)^2\right)^{3/2}} = \frac{|2e^x + xe^x|}{\left(1 + (e^x + xe^x)^2\right)^{3/2}}.$$

Alternatively, we could use the formula $\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|}$, but doing so will be more complicated. Alternatively, the book has the formula

$$\kappa = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{3/2}}.$$

If you wrote down this formula on your note card, you could obtain the above answer slightly more quickly.

7. Find the velocity, acceleration, and speed of a particle with the position function

$$\mathbf{r}(t) = t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \sin t\,\mathbf{k}.$$

Solution: We compute that

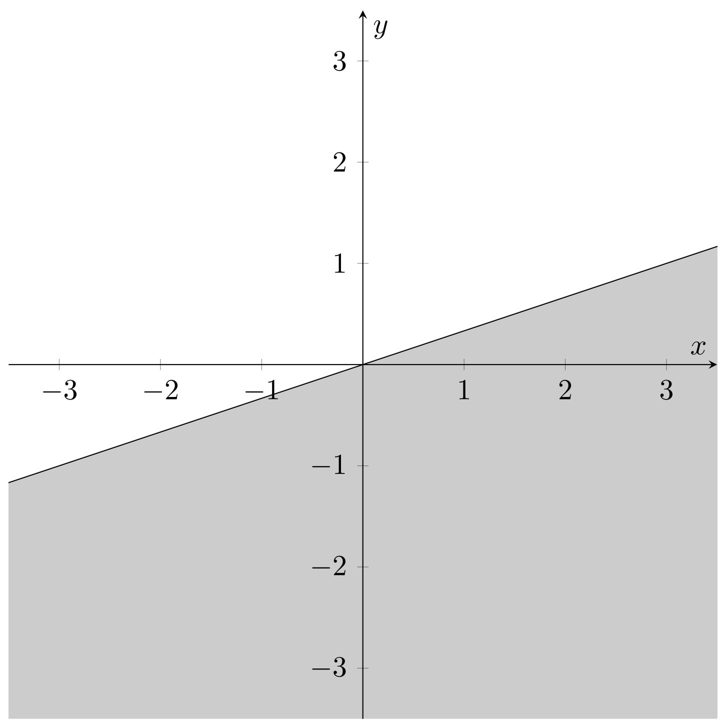
$$\begin{aligned} \mathbf{v}(t) &= \mathbf{r}'(t) = \mathbf{i} - 2\sin t\,\mathbf{j} + \cos t\,\mathbf{k}, \\ \mathbf{a}(t) &= \mathbf{v}'(t) = -2\cos t\,\mathbf{j} - \sin t\,\mathbf{k}, \\ v(t) &= |\mathbf{v}(t)| = \sqrt{1 + 4\sin^2 t + \cos^2 t} = \sqrt{2 + 3\sin^2 t}. \end{aligned}$$

8. Find and sketch the domain of the function

$$f(x,y) = \sqrt[4]{x-3y}.$$

Solution: We can only take the fourth root of a nonnegative number, so the domain is all points (x, y) such that $x - 3y \geq 0$.

To sketch this domain, we draw the line $x - 3y = 0$, which some of you might find it more familiar to rewrite as $y = \frac{1}{3}x$. This line contains the points where $x - 3y = 0$. The region $x - 3y \geq 0$ has points with larger x-values than points on this line, so we shade in the region to the right of the line.



9. Use polar coordinates to find the limit

$$\lim_{(x,y)\to(0,0)} \left(x^2+y^2\right) \ln\left(x^2+y^2\right).$$

Solution: To use polar coordinates, we substitute r^2 for x^2+y^2, and we note that $(x, y) \to 0$ is the same as $r \to 0^+$. We compute using L'Hôpital's rule.

$$\lim_{(x,y)\to(0,0)} \left(x^2+y^2\right) \ln\left(x^2+y^2\right) = \lim_{r\to 0^+} r^2 \ln\left(r^2\right)$$
$$= \lim_{r\to 0^+} \frac{\ln(r^2)}{r^{-2}} = \lim_{r\to 0^+} \frac{\frac{2r}{r^2}}{-2r^{-3}} = \lim_{r\to 0^+} -r^2 = 0.$$

10. Let

$$f(x, y) = x^2 y - 3y^4.$$

Find f_x and f_y.

Solution: We compute

$$f_x(x, y) = 2xy - 0 = 2xy,$$
$$f_y(x, y) = x^2 - 12y^3.$$

11. Find an equation of the tangent plane to the surface

$$z = e^{x-y}$$

at the point $(2, 2, 1)$.

Solution: We compute

$$\frac{\partial z}{\partial x} = e^{x-y}, \qquad \frac{\partial z}{\partial x}(2,2) = 1,$$
$$\frac{\partial z}{\partial y} = -e^{x-y}, \qquad \frac{\partial z}{\partial y}(2,2) = -1.$$

The equation of the tangent plane at $(2, 2, 1)$ is

$$\begin{aligned} z - 1 &= \frac{\partial z}{\partial x}(x-2) + \frac{\partial z}{\partial y}(y-2), \\ z &= (x-2) - (y-2) + 1, \\ z &= x - y + 1. \end{aligned}$$

Alternatively, we let $F(x, y, z) = e^{x-y} - z$, so

$$\frac{\partial F}{\partial x} = e^{x-y}, \qquad \frac{\partial F}{\partial x}(2,2,1) = 1,$$
$$\frac{\partial F}{\partial y} = -e^{x-y}, \qquad \frac{\partial F}{\partial y}(2,2,1) = -1,$$
$$\frac{\partial F}{\partial z} = -1, \qquad \frac{\partial F}{\partial z}(2,2,1) = -1.$$

Thus, $\langle 1, -1, -1 \rangle$ is the normal vector to the plane. The equation of the plane through $(2, 2, 1)$ with this normal vector is

$$\begin{aligned} (x-2) - (y-2) - (z-1) &= 0, \\ x - y - z &= -1. \end{aligned}$$

12. Let

$$z = x^4 + x^2 y, \qquad x = s - 2t - u, \qquad y = stu^2.$$

Compute $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$, and $\frac{\partial z}{\partial u}$ when $s = 4$, $t = 2$, and $u = 1$.

Solution: We compute

$$\frac{\partial z}{\partial x} = 4x^3 + 2xy, \qquad \frac{\partial z}{\partial y} = x^2.$$

and

$$\frac{\partial x}{\partial s} = 1, \qquad \frac{\partial x}{\partial t} = -2, \qquad \frac{\partial x}{\partial u} = -1,$$
$$\frac{\partial y}{\partial s} = tu^2, \qquad \frac{\partial y}{\partial t} = su^2, \qquad \frac{\partial y}{\partial u} = 2stu.$$

When $s = 4$, $t = 2$, and $u = 1$, we compute that that $x = -1$ and $y = 8$. Evaluating the partial derivatives, we find that

$$\frac{\partial z}{\partial x} = -4 - 2(8) = -20, \qquad \frac{\partial z}{\partial y} = 1.$$

and

$$\frac{\partial x}{\partial s} = 1, \qquad \frac{\partial x}{\partial t} = -2, \qquad \frac{\partial x}{\partial u} = -1,$$
$$\frac{\partial y}{\partial s} = 2, \qquad \frac{\partial y}{\partial t} = 4, \qquad \frac{\partial y}{\partial u} = 16.$$

Finally, we compute using the chain rule that

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (-20)(1) + (1)(2) = -18,$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (-20)(-2) + (1)(4) = 44,$$
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (-20)(-1) + (1)(16) = 36.$$

13. Find the equation of the tangent plane to the hyperboloid

$$xy + yz + zx = 5$$

at the point $(1, 2, 1)$.

Solution: This surface is a level set of the function $F(x, y, z) = xy + yz + zx$. Thus, the gradient of F points away from the surface. We compute that

$$\begin{aligned} \nabla F &= \langle y + z, x + z, y + x \rangle, \\ \nabla F(1, 2, 1) &= \langle 3, 2, 3 \rangle. \end{aligned}$$

The vector $\langle 3, 2, 3 \rangle$ points away from the surface, so it is perpendicular to the tangent plane. The equation of the tangent plane through $(1, 2, 1)$ with normal vector $\langle 3, 2, 3 \rangle$ is

$$\begin{aligned} 3(x - 1) + 2(y - 2) + 3(z - 1) &= 0, \\ 3x + 2y + 3z &= 10. \end{aligned}$$

14. Find the points on the hyperboloid $y^2 = 9 + xz$ that are closest to the origin.

Solution: As we saw on Webassign, it's easier to work with the distance squared rather than the distance itself. The distance squared between a point (x, y, z) and the origin is $x^2 + y^2 + z^2$. For a point on the surface, we have $y^2 = 9 + xz$. Substituting, we see that the distance squared between the origin a point (x, y, z) on the surface is $x^2 + (9 + xz) + z^2$. We would like to minimize this function.

Setting $f(x, z) = x^2 + 9 + xz + z^2$, we seek the critical points of f. We compute

$$\frac{\partial f}{\partial x} = 2x + z, \qquad \frac{\partial f}{\partial z} = x + 2z.$$

We can check that $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial z} = 0$ whenever $x = z = 0$. This point is our only critical point, so the distance to the origin is minimized when $x = z = 0$. When $x = z = 0$, we have that $y^2 = 9 + 0$, so $y = \pm 3$. Thus, the two points on the surface that are closest to the origin are $(0, 3, 0)$ and $(0, -3, 0)$.