Math 308 Midterm 2

April 4, 2018

Name: ______________________________________

- Show your work. If you solve a problem with anything other than a straightforward computation, write one complete sentence explaining what you're doing.
  - For example, if you're computing a cross product using the standard method, just show your computation.
  - But if, for example, you find that a line integral is zero without actually computing the line integral, you need to write one complete sentence convincing an imaginary peer that that's true.
- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
  - Make a note on the printed page that your work continues on the back of the previous page.
  - On the back of the previous page, put a box around the work that you want graded.
- There are four questions, worth between 15 and 35 points each.
  - The problems are ordered by topic, not by difficulty.

1. Consider the ball of radius 3 centered around the origin. The electric displacement field is

$$\mathbf{D} = x\mathbf{i} + y\mathbf{j} + \left(z + x^2 + y^2 + z^2 - 9\right)\mathbf{k}.$$

The units of x, y, and z are in meters, and the units of $\mathbf{D}$ are Coulombs/meter^2. In this problem, you will compute the total charge inside the ball in two different ways.

(a) (10 points) Compute the total charge inside the ball by evaluating an appropriate surface integral on the sphere of radius 3. Specify the units of your answer.

Solution: Using Gauss's law, we need to compute

$$\oint_{\text{sphere}} \mathbf{D} \cdot \mathbf{n}\, d\sigma.$$

The equation of the sphere of radius 3 is $x^2 + y^2 + z^2 = 9$. Thus, on the sphere, we have

$$\begin{aligned}
\oint_{\text{sphere}} \mathbf{D} \cdot \mathbf{n}\, d\sigma &= \oint_{\text{sphere}} \left(x\mathbf{i} + y\mathbf{j} + \left(z + x^2 + y^2 + z^2 - 9\right)\mathbf{k}\right) \cdot \mathbf{n}\, d\sigma \\
&= \oint_{\text{sphere}} \left(x\mathbf{i} + y\mathbf{j} + (z + 9 - 9)\mathbf{k}\right) \cdot \mathbf{n}\, d\sigma \\
&= \oint_{\text{sphere}} \left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right) \cdot \mathbf{n}\, d\sigma
\end{aligned}$$

We've seen that the vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ points outwards. It has size 3 on the sphere of radius 3. We can use this information to compute $\mathbf{n}$, or we can just note that $\mathbf{r} \cdot \mathbf{n} = 3$ on the sphere of radius 3.
Thus, the total charge inside the ball is

$$\oint_{\text{sphere}} \mathbf{D} \cdot \mathbf{n}\, d\sigma = \oint_{\text{sphere}} 3\, d\sigma = 4\pi(3)^2 \cdot 3 = \boxed{108\pi},$$

in Coulombs. (We can determine that the units are Coulombs by noting that we are integrating a Coulombs/meter^2 quantity over an area expressed in meters^2, giving us an answer in Coulombs.)

(b) (10 points) Compute the total charge inside the ball by evaluating an appropriate volume integral on the ball of radius 3. Specify the units of your answer.

Solution: Using the divergence theorem, we need to compute

$$\int_{\text{ball}} \operatorname{div} \mathbf{D}\, dx\, dy\, dz.$$

We compute that

$$\int_{\text{ball}} \operatorname{div} \mathbf{D}\, dx\, dy\, dz = \int_{\text{ball}} (1 + 1 + (1 + 2z))\; dx\, dy\, dz = \int_{\text{ball}} (3 + 2z)\; dx\, dy\, dz.$$

We see that

$$\int_{\text{ball}} 3\,dx\,dy\,dz = \tfrac{4}{3}\pi(3)^3 \cdot 3 = 108\pi.$$

Meanwhile, we can see that

$$\int_{\text{ball}} 2z\,dx\,dy\,dz = 0$$

by symmetry. Because the ball is symmetric and because $2z$ is an odd function of z, it's clear that the integral of $2z$ over the lower half of the ball where $z < 0$ will exactly cancel out the integral of $2z$ over the upper half of the ball where $z > 0$.

Thus, the total charge inside the ball is

$$\int_{\text{ball}} \operatorname{div} \mathbf{D}\,dx\,dy\,dz = \boxed{108\pi},$$

in Coulombs.

2. Consider the surface S of the horizontal cylinder of radius 3 and length 10, described by the equations $y^2 + z^2 = 9$ and $-5 \le x \le 5$. Let

$$\mathbf{A} = -xz\mathbf{j} + xy\mathbf{k}.$$

You can compute that

$$\operatorname{curl} \mathbf{A} = 2x\mathbf{i} - y\mathbf{j} - z\mathbf{k}.$$

The two ends of the cylinder are closed circular curves. Call them γ_- and γ_+, oriented as shown. Each curve is the boundary of a disk. Call those disks D_- and D_+, and orient them using unit normals as shown. Call the round surface of the cylinder S, and orient it as shown.

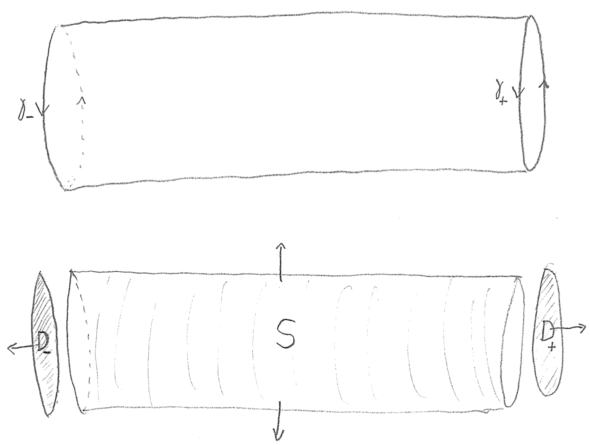

Figure 1: Please ask me if you have questions about interpreting this schematic diagram.

Be careful about orientations and signs in this problem.

(a) (15 points) Compute $\int_{D_-} \operatorname{curl} \mathbf{A} \cdot \mathbf{n}\, d\sigma$ and $\int_{D_+} \operatorname{curl} \mathbf{A} \cdot \mathbf{n}\, d\sigma$ and then explain what that tells you about $\oint_{\gamma_-} \mathbf{A} \cdot d\mathbf{r}$ and $\oint_{\gamma_+} \mathbf{A} \cdot d\mathbf{r}$.

Solution: The disk D_- has equation $x = -5$, $y^2 + z^2 \le 9$. We see that for this surface, $\mathbf{n} = -\mathbf{i}$, so

$$\operatorname{curl} \mathbf{A} \cdot \mathbf{n} = -2x = (-2)(-5) = 10.$$

The disk has area $\pi(3)^2 = 9\pi$, so the surface integral is

$$\int_{D_-} \operatorname{curl} \mathbf{A} \cdot \mathbf{n}\, d\sigma = \boxed{90\pi}.$$

We see that the orientation of D_- is incompatible with the orientation of its boundary γ_-, so we need to flip a sign and conclude that

$$\oint_{\gamma_-} \mathbf{A} \cdot d\mathbf{r} = \boxed{-90\pi}.$$

For D_+, the story is the same, except we have $x = 5$ instead of -5, and we have $\mathbf{n} = \mathbf{i}$ instead of $-\mathbf{i}$. We see that these two sign changes cancel, so

$$\int_{D_+} \operatorname{curl} \mathbf{A} \cdot \mathbf{n}\, d\sigma = \boxed{90\pi}.$$

This time, however, the orientation of D_+ is compatible with γ_+, so

$$\oint_{\gamma_+} \mathbf{A} \cdot d\mathbf{r} = \boxed{90\pi}.$$

(b) (15 points) Compute $\int_S \operatorname{curl} \mathbf{A} \cdot \mathbf{n}\, d\sigma$ and then explain what that tells you about $\oint_{\gamma_-} \mathbf{A} \cdot d\mathbf{r}$ and $\oint_{\gamma_+} \mathbf{A} \cdot d\mathbf{r}$.

Solution: The round surface of the cylinder has the equation given in the problem, $y^2 + z^2 = 9$ and $-5 \le x \le 5$. The vector $y\mathbf{j} + z\mathbf{k}$ points in the normal direction, but it has size $\sqrt{y^2 + z^2} = 3$, so the unit normal is $\mathbf{n} = \frac{1}{3}(y\mathbf{j} + z\mathbf{k})$. We can use this fact to compute that

$$\operatorname{curl} \mathbf{A} \cdot \mathbf{n} = \frac{1}{3}(y^2 + z^2) = -3.$$

Alternatively, we observe that the $\mathbf{i}$ direction is tangent to the cylinder S, so the $2x\mathbf{i}$ part is irrelevant. Meanwhile, $-y\mathbf{j} - z\mathbf{k}$ is normal to S, pointing inward, with size $y^2 + z^2 = 3$. Thus, again, we find that $\operatorname{curl} \mathbf{A} \cdot \mathbf{n} = -3$.

The area of the cylinder is $2\pi(3)(10) = 60\pi$, so

$$\int_S \operatorname{curl} \mathbf{A} \cdot \mathbf{n}\, d\sigma = \boxed{-180\pi}.$$

Being careful with orientations, we see that $\partial S = \gamma_- - \gamma_+$, so our computation lets us conclude that

$$\boxed{\oint_{\gamma_-} \mathbf{A} \cdot d\mathbf{r} - \oint_{\gamma_+} \mathbf{A} \cdot d\mathbf{r} = -180\pi.}$$

Equivalently,

$$\boxed{\oint_{\gamma_+} \mathbf{A} \cdot d\mathbf{r} - \oint_{\gamma_-} \mathbf{A} \cdot d\mathbf{r} = 180\pi.}$$

The computation in this part does not let us compute these line integrals individually; we can only find their difference. Reassuringly, our result is consistent with the values for the line integrals that we found in the previous part.

(c) (0 points) For up to 5 bonus points, check your work by computing the line integrals $\oint_{\gamma_-} \mathbf{A} \cdot d\mathbf{r}$ and $\oint_{\gamma_+} \mathbf{A} \cdot d\mathbf{r}$ directly.

Solution: Starting with γ_+, we see that it can be described with the equation $x = 5$, $y^2 + z^2 = 9$. To get the right answer, we must parametrize this curve in a way that's compatible with its orientation; that is, we must travel in the right direction. One way to do so is

$$\begin{aligned} x &= 5 & dx &= 0 \\ y &= 3\cos t & dy &= -3\sin t\, dt \\ z &= 3\sin t & dz &= 3\cos t\, dt \end{aligned}$$

for $0 \le t \le 2\pi$. To check that the orientation is correct, observe that when $z = 0$ and $y = 3$, that is, we are on the "far" side of the cylinder, $\frac{dz}{dt} > 0$, so we move upwards, which is consistent with the orientation in the diagram.

Thus

$$\begin{aligned}\oint_{\gamma_+} \mathbf{A}\cdot d\mathbf{r} &= \int_0^{2\pi} -(5)(3\sin t)(-3\sin t\,dt) + (5)(3\cos t)(3\cos t\,dt)\\ &= \int_0^{2\pi} 45(\cos^2 t + \sin^2 t)\,dt\\ &= (2\pi)(45)(1) = \boxed{90\pi}.\end{aligned}$$

For γ_-, we see that the only thing that changes is that we have $x = -5$ instead of 5. Other than that change, the same parametrization works, and it is still consistent with the orientation of γ_- in the diagram. The two curves are just translated versions of each other; up to changing the x value, they are the same.

Looking through our line integral computation, we see that changing 5 to -5 simply flips the signs of all the terms, which is unsurprising since changing x to $-x$ simply flips the signs of $\mathbf{A}$. Thus,

$$\oint_{\gamma_-} \mathbf{A}\cdot d\mathbf{r} = \boxed{-90\pi}.$$

Again, reassuringly, our answer is consistent with what we had before.

3. In the previous problem, you worked with the vector fields

$$\begin{aligned}\mathbf{A} &= -xz\mathbf{j} + xy\mathbf{k},\\ \mathbf{V} &= 2x\mathbf{i} - y\mathbf{j} - z\mathbf{k},\end{aligned}$$

satisfying $\operatorname{curl}\mathbf{A} = \mathbf{V}$.

(a) (10 points) Find a vector field $\mathbf{A}'$ such that $\operatorname{curl}\mathbf{A}' = \mathbf{V}$ and such that $A'_y = e^x$.

Solution: In class, we saw that, absent any unfortunate holes in the domain, any two vector fields with the same curl differ by a gradient and vice versa; that is $\mathbf{A}' = \mathbf{A} + \operatorname{grad} u$ for some u. Looking at the $\mathbf{j}$ component of this equation, we see that we need a u so that

$$\begin{aligned}e^x &= -xz + \frac{\partial u}{\partial y},\\ \frac{\partial u}{\partial y} &= e^x + xz.\end{aligned}$$

We see that $u = e^x y + xyz$ sufficies, though there are many other valid choices. For this choice of u, we find that

$$\begin{aligned}\mathbf{A}' &= \mathbf{A} + \operatorname{grad} u\\ &= (-xz\mathbf{j} + xy\mathbf{k}) + ((e^x y + yz)\mathbf{i} + (e^x + xz)\mathbf{j} + xy\mathbf{k})\\ &= (e^x y + yz)\mathbf{i} + e^x\mathbf{j} + 2xy\mathbf{k}.\end{aligned}$$

(b) (5 points) If γ is a closed curve, do you expect $\oint_\gamma \mathbf{A}\cdot d\mathbf{r}$ and $\oint_\gamma \mathbf{A}'\cdot d\mathbf{r}$ to be equal? Why or why not?

Solution: We know that the line integral around a closed curve of a gradient is zero, either from the perspective that its start point is the same as its end point or from the perspective that it has no boundary. That is, $\oint_\gamma \operatorname{grad} u \cdot d\mathbf{r} = 0$ for any u. Thus,

$$\oint_\gamma \mathbf{A}'\cdot d\mathbf{r} = \oint_\gamma \mathbf{A}\cdot d\mathbf{r} + \oint_\gamma \operatorname{grad} u\cdot d\mathbf{r} = \oint_\gamma \mathbf{A}\cdot d\mathbf{r}.$$

4. (a) (25 points) Let $f(x)$ be the 2π-periodic function such that

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < \frac{2\pi}{3}, \\ 0 & \text{for } \frac{2\pi}{3} < x < 2\pi. \end{cases}$$

Compute the Fourier series for f.

Guidelines for acceptable answers by way of analogy:

- Acceptable: $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.
- Acceptable: $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$. Include enough terms so the pattern is clear.
- Unacceptable: $c_n = \frac{1}{n!}$.
- Acceptable: $e^x = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$, where $c_n = \frac{1}{n!}$.

Solution: As usual, we set

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right).$$

We compute that

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) 1 \, dx \\ &= \frac{1}{\pi} \int_0^{2\pi/3} 1 \, dx \\ &= \frac{1}{\pi} \cdot \frac{2\pi}{3} = \frac{2}{3}. \end{aligned}$$

We then compute that, for $n \neq 0$.

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\ &= \frac{1}{\pi} \int_0^{2\pi/3} \cos nx \, dx \\ &= \frac{1}{\pi n} \left. (\sin nx) \right|_0^{2\pi/3} \\ &= \frac{1}{\pi n} \sin(2\pi n/3). \end{aligned}$$

When n is divisible by 3, $\sin(2\pi n/3) = \sin 0 = 0$. When n is 1 more than a multiple of 3, $\sin(2\pi n/3) = \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$. When n is 2 more than a multiple of 3, $\sin(2\pi n/3) = \sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$. Thus,

$$a_n = \begin{cases} \frac{2}{3} & \text{if } n = 0, \\ 0 & \text{if } n \text{ is a nonzero multiple of 3}, \\ \frac{\sqrt{3}}{2\pi n} & \text{if } n \text{ is 1 more than a multiple of 3}, \\ -\frac{\sqrt{3}}{2\pi n} & \text{if } n \text{ is 2 more than a multiple of 3}. \end{cases}$$

Similarly, we compute that

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \\ &= \frac{1}{\pi}\int_0^{2\pi/3} \sin nx\,dx \\ &= -\frac{1}{\pi n}\left(\cos nx\right)\Big|_0^{2\pi/3} \\ &= \frac{1}{\pi n}\left(1-\cos(2\pi n/3)\right). \end{aligned}$$

When n is divisible by 3, $\cos(2\pi n/3) = \cos 0 = 1$. When n is 1 more than a multiple of 3, $\cos(2\pi n/3) = \cos(2\pi/3) = -\frac{1}{2}$. When n is 2 more than a multiple of 3, $\cos(2\pi n/3) = \cos(4\pi/3) = -\frac{1}{2}$. Thus,

$$b_n = \begin{cases} 0 & \text{if } n \text{ is a multiple of 3,} \\ \frac{3}{2\pi n} & \text{otherwise.} \end{cases}$$

Thus,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right),$$

where a_n and b_n are as above.

(b) (10 points) Compute the sum of the series that you found in part (a) for each of the following five values of x: $0, \frac{\pi}{3}, \frac{2\pi}{3}, 2\pi$, and 3π.

Your answer for each value of x should be a number. It is neither sufficient nor required to write down a series.

Solution: We know from class that the sum of the series is equal to $f(x)$ when f is continuous and equal to the midpoint of the jump in f when f has a jump discontinuity. Drawing a graph of the function if needed, we can determine these values.

- At $x = 0$, the function jumps from 0 to 1, so the sum of the series is $\boxed{\frac{1}{2}}$ when $x = 0$.
- At $x = \frac{\pi}{3}$, the function is continuous, so the sum of the series is $f(\frac{\pi}{3}) = \boxed{1}$.
- At $x = \frac{2\pi}{3}$, the function jumps from 1 to 0, so the sum of the series is $\boxed{\frac{1}{2}}$.
- At $x = 2\pi$, we can apply the same reasoning, or we can note that we have the same situation as at $x = 0$ by 2π-periodicity, so the sum of the series is $\boxed{\frac{1}{2}}$.
- At $x = 3\pi$, by 2π-periodicity, we get the same answer as at $x = \pi$. At $x = \pi$, the function is continuous, so the sum of the series is $f(\pi) = \boxed{0}$.

Question	Points	Score
1	20	
2	30	
3	15	
4	35	
Total:	100	