Math 308 Final

May 3, 2018

Name: ____

- Show your work. If you solve a problem with anything other than a straightforward computation, write one complete sentence explaining what you're doing.
  - For example, if you're computing a cross product using the standard method, just show your computation.
  - But if, for example, you find that a line integral is zero without actually computing the line integral, you need to write one complete sentence convincing an imaginary peer that that's true.
- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
  - Make a note on the printed page that your work continues on the back of the previous page.
  - On the back of the previous page, put a box around the work that you want graded.
- There are 10 questions, worth between 5 and 25 points each.
  - The problems are ordered by topic, not by difficulty.
- Guidelines for acceptable answers for Fourier series questions by way of analogy:
  - Acceptable: $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.
  - Acceptable: $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$. Include enough terms so the pattern is clear.
  - Unacceptable: $c_n = \frac{1}{n!}$.
  - Acceptable: $e^x = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$, where $c_n = \frac{1}{n!}$.
  - Acceptable: $e^x = \sum_{n=0}^{\infty} c_n x^n$, where $c_n = \frac{1}{n!}$.

1. (5 points) According to my phone, I am currently at a latitude of $38.6487°N$ and a longtitude of $90.3057°W$ at an altitude of about 500 feet above sea level. Using an online tool to convert this position to rectangular coordinates while taking into account the fact that the Earth is not perfectly spherical, my position vector is approximately

$$-(5000\,\text{km})\mathbf{j} + (4000\,\text{km})\mathbf{k}.$$

What is my velocity vector with respect to the center of the Earth? Include units.

Facts, some of which are helpful:

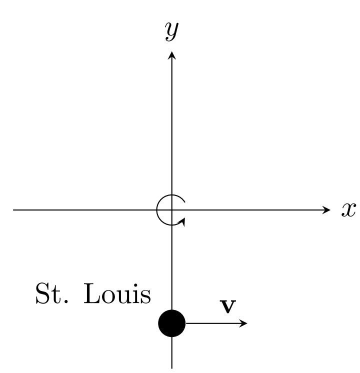
- The standard rectangular coordinates are centered on the center of the Earth, with the z-direction pointing along the Earth's axis of rotation towards the North pole, and with Greenwich, England in the xz-plane with a positive x-value.
- The Earth takes about 23 hours and 56 minutes to complete a full rotation, going counter-clockwise from the persepctive of someone looking down on the North pole. (The extra 4 minutes of the day come from the motion of the Earth around the Sun.) For the purposes of this problem, approximate this rotation rate as 0.3 radians/hour.

Solution: In the formula $\mathbf{v} = \omega \times \mathbf{r}$, based on the rotation axis and direction, we see that $\omega = (0.3\,\text{radians/hour})\mathbf{k}$. Since $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$ and $\mathbf{k} \times \mathbf{k} = 0$, we see that

$$\mathbf{v} = (0.3\,\text{radians/hour})\mathbf{k} \times (-(5000\,\text{km})\mathbf{j} + (4000\,\text{km})\mathbf{k}) = (1500\,\text{km/hr})\mathbf{i},$$

since radians are unitless.

We can check our signs with a picture looking down from above.



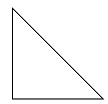
2. Consider the triangle in the xy-plane with vertices $(0,0)$, $(6,0)$, and $(0,6)$. Consider the vector field

$$\mathbf{F} = -y\mathbf{i} + x\mathbf{j}.$$

Let γ be the closed path going around the triangle counter-clockwise.

(a) (5 points) Compute $\oint_\gamma \mathbf{F} \cdot d\mathbf{r}$ directly, without using Stokes' (Green's) Theorem.

Solution:



We first compute that

$$\mathbf{F} \cdot d\mathbf{r} = -y\,dx + x\,dy.$$

The left side of the triangle has the equation $x = 0$, so $dx = 0$, and so

$$\mathbf{F} \cdot d\mathbf{r} = -y(0) + (0)\,dy = 0$$

on that side. Similarly, the bottom side of the triangle has $y = 0$, so $dy = 0$, and so $\mathbf{F} \cdot d\mathbf{r} = 0$ there as well.

Finally, we come to the slanted side of the triangle. Because the curve is oriented counter-clockwise, we move up and to the left along the slanted side, so a reasonable parametrization is

$$x = 6 - t, \qquad\qquad y = t,$$

for $0 \leq t \leq 6$. Hence

$$dx = -dt, \qquad\qquad dy = dt,$$

and so, along this side

$$\mathbf{F} \cdot d\mathbf{r} = -(t)(-dt) + (6-t)(dt) = t\,dt + 6\,dt - t\,dt = 6\,dt.$$

Since this side is the only side that contributes to the line integral, we find that

$$\oint_\gamma \mathbf{F} \cdot d\mathbf{r} = \int_0^6 6\,dt = 36.$$

(b) (5 points) Compute $\oint_\gamma \mathbf{F} \cdot d\mathbf{r}$ using Stokes' (Green's) Theorem.

Solution: Stokes' Theorem tells us that

$$\oint_\gamma \mathbf{F} \cdot d\mathbf{r} = \iint_{\text{triangle}} \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\,d\sigma = \iint_{\text{triangle}} \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\,dx\,dy.$$

We compute that

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathbf{k} = (1 - (-1))\mathbf{k} = 2\mathbf{k}.$$

Therefore,

$$\oint_\gamma \mathbf{F} \cdot d\mathbf{r} = \iint_{\text{triangle}} 2\, dx\, dy = 2 \cdot \frac{1}{2}(6)(6) = 36.$$

3. (5 points) Consider the three vector fields below.

$$\mathbf{V}_1 = (2x + y + z)\mathbf{i} + (x + y + 2z)\mathbf{j} + (x - 2y - z)\mathbf{k}.$$
$$\mathbf{V}_2 = (x - 2y - z)\mathbf{i} + (2x + y + z)\mathbf{j} + (x + y + 2z)\mathbf{k}.$$
$$\mathbf{V}_3 = (x + y + 2z)\mathbf{i} + (x - 2y - z)\mathbf{j} + (2x + y + z)\mathbf{k}.$$

Which of these three vector fields has a vector potential? That is, which of these three vector fields is equal to the curl of another vector field?

For a complete solution, you should explain

- why one of the vector fields has a vector potential, and
- why the two other vector fields do not have a vector potential.

Finding a vector potential certainly counts as a valid explanation for the first bullet point, but doing so is not required.

Solution: Since $\operatorname{div}\operatorname{curl}\mathbf{A} = 0$, if $\mathbf{V} = \operatorname{curl}\mathbf{A}$, then $\operatorname{div}\mathbf{V} = 0$. Meanwhile, since the domain has no holes, if $\operatorname{div}\mathbf{V} = 0$, there must exist a vector potential. We compute the divergence of the three vector fields.

$$\operatorname{div}\mathbf{V}_1 = 2 + 1 - 1 = 2.$$
$$\operatorname{div}\mathbf{V}_2 = 1 + 1 + 2 = 4.$$
$$\operatorname{div}\mathbf{V}_3 = 1 - 2 + 1 = 0.$$

Thus, $\mathbf{V}_3$ has a vector potential, but $\mathbf{V}_1$ and $\mathbf{V}_2$ do not.

4. Consider the 2π-periodic function defined by

$$f(x) = \begin{cases} 7 & \text{if } 0 < x < \pi \\ 0 & \text{if } \pi < x < 2\pi. \end{cases}$$

Be sure to follow the guidelines for acceptable answers on the exam cover.

(a) (10 points) Find the Fourier series of f in terms of sines and cosines. Compute directly, without using your answer to part (b).

Solution: Our Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

Using the formulas for the coefficients, we find that

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} 7\,dx = 7. \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} 7\cos(nx)\,dx \\
&= \frac{7}{\pi}\left(\tfrac{1}{n}\right)(\sin(nx))\Big|_0^{\pi} \\
&= \frac{7\pi}{n}(\sin(n\pi) - \sin 0) = 0. \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} 7\sin(nx)\,dx \\
&= \frac{7}{\pi}\left(\tfrac{1}{n}\right)(-\cos(nx))\Big|_0^{\pi} \\
&= \frac{7}{\pi n}(-\cos(n\pi) + 1) \\
&= \frac{7}{\pi n}\begin{cases} 2 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \\
&= \begin{cases} \frac{14}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.
\end{aligned}$$

Thus, the Fourier series for f is

$$f(x) = \frac{7}{2} + \frac{14}{\pi}\sum_{\substack{n \text{ odd} \\ n>0}} \frac{1}{n}\sin(nx)$$

(b) (5 points) Find the Fourier series of f in terms of complex exponentials. Compute directly, without using your answer to part (a).

Solution: In this case, we write

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

To compute the coefficients, if $n \neq 0$, we compute

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx} \\ &= \frac{1}{2\pi}\int_0^{\pi} 7e^{-inx} \\ &= \frac{1}{2\pi}\frac{1}{-in}\, 7e^{-in\frac{\pi}{3}x}\Big|_0^{\pi} \\ &= \frac{7}{2\pi in}\left(-e^{-in\pi} + e^0\right) \\ &= \frac{7}{2\pi in}\begin{cases} 2 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \\ &= \begin{cases} \frac{7}{\pi in} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

Meanwhile, when $n = 0$, we have that

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{0x} = \frac{1}{2\pi}\int_0^{\pi} 7 = \frac{7}{2}.$$

Thus,

$$f(x) = \frac{7}{2} + \frac{7}{\pi i}\sum_{n \text{ odd}} \frac{1}{n}e^{inx}.$$

(c) (5 points) Show that your answers in parts (a) and (b) are equal.

Solution: We can start from either answer and work our way to the other one. Starting from our answer to part (b), we find that

$$\begin{aligned}
\frac{7}{2} + \frac{7}{\pi i} \sum_{n \text{ odd}} \frac{1}{n} e^{inx} &= \frac{7}{2} + \frac{7}{\pi i} \left(\sum_{\substack{n \text{ odd} \\ n>0}} \frac{1}{n} e^{inx} + \sum_{\substack{n \text{ odd} \\ n<0}} \frac{1}{n} e^{inx} \right) \\
&= \frac{7}{2} + \frac{7}{\pi i} \left(\sum_{\substack{n \text{ odd} \\ n>0}} \frac{1}{n} e^{inx} + \sum_{\substack{n \text{ odd} \\ n>0}} \frac{1}{-n} e^{-inx} \right) \\
&= \frac{7}{2} + \frac{7}{\pi i} \sum_{\substack{n \text{ odd} \\ n>0}} \frac{1}{n} \left(e^{inx} - e^{inx} \right) \\
&= \frac{7}{2} + \frac{7}{\pi i} \sum_{\substack{n \text{ odd} \\ n>0}} \frac{1}{n} \left(2i \sin(nx) \right) \\
&= \frac{7}{2} + \frac{14}{\pi} \sum_{\substack{n \text{ odd} \\ n>0}} \frac{1}{n} \sin nx,
\end{aligned}$$

which is our answer to part (a).

(d) (5 points) Apply Parseval's Theorem to either the series in part (a) or part (b). Doing both is not required but will let you check your work. Your final answer should be an equation with a number on one side and a series on the other side.

Solution: Parseval's Theorem states that

$$\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |c_n|^2 .$$

The left-hand side is

$$\frac{1}{2\pi} \int_0^{\pi} 7^2 \, dx = \frac{49}{2}.$$

The right-hand side is

$$\left(\frac{7}{2}\right)^2 + \sum_{n \text{ odd}} \left| \frac{7}{\pi i n} \right|^2 = \frac{49}{4} + \sum_{n \text{ odd}} \frac{49}{\pi^2 n^2}.$$

Thus Parseval's Theorem states that

$$\frac{49}{2} = \frac{49}{4} + \sum_{n \text{ odd}} \frac{49}{\pi^2 n^2}.$$

We can, optionally, simplify this expression.

$$\frac{1}{2} = \frac{1}{4} + \sum_{n \text{ odd}} \frac{1}{\pi^2 n^2}$$
$$\frac{1}{4} = \sum_{n \text{ odd}} \frac{1}{\pi^2 n^2}$$
$$\frac{1}{4} = 2 \sum_{\substack{n \text{ odd} \\ n>0}} \frac{1}{\pi^2 n^2}$$
$$\frac{\pi^2}{8} = \sum_{\substack{n \text{ odd} \\ n>0}} \frac{1}{n^2}.$$

5. (5 points) Consider the function f on the whole real line defined by

$$f(x) = \begin{cases} 17 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Compute the Fourier transform of f. The guidelines for acceptable answers for Fourier series apply here as well, except you'll have an integral instead of a sum.

Solution: We compute that

$$\begin{aligned} g(\alpha) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x}\, dx \\ &= \frac{1}{2\pi} \int_{0}^{1} 17 e^{-i\alpha x}\, dx \\ &= \frac{17}{2\pi} \frac{1}{-i\alpha}\, e^{-i\alpha x}\Big|_0^1 \\ &= \frac{17}{2\pi} \frac{1}{i\alpha} \left(-e^{-i\alpha} + e^0\right) \\ &= \frac{17}{2\pi i} \frac{1}{\alpha} \left(1 - e^{-i\alpha}\right). \end{aligned}$$

Thus,

$$f(x) = \int_{-\infty}^{\infty} g(\alpha) e^{i\alpha x}\, d\alpha,$$

where $g(\alpha) = \frac{17}{2\pi i}\frac{1}{\alpha}\left(1 - e^{-i\alpha}\right)$.

6. Consider an infinite rectangular plate of width 2π, so $0 \le x \le 2\pi$ and $0 \le y < \infty$. The left and right sides of the rectangular plate are held at 0 degrees. The bottom side is held at $f(x)$, where

$$f(x) = \begin{cases} 100 & \text{if } 0 < x < \pi, \\ -100 & \text{if } \pi < x < 2\pi. \end{cases}$$

Helpful facts:

- The following functions T_n are solutions to the equation $\Delta T_n = 0$.

$$T_n(x, y) = \sin(k_n x) e^{-k_n y},$$

where $k_n = \frac{n}{2}$, and n is a positive integer.

- The sine series for $f(x)$ is

$$f(x) = \frac{400}{\pi}\left(\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \cdots\right)$$

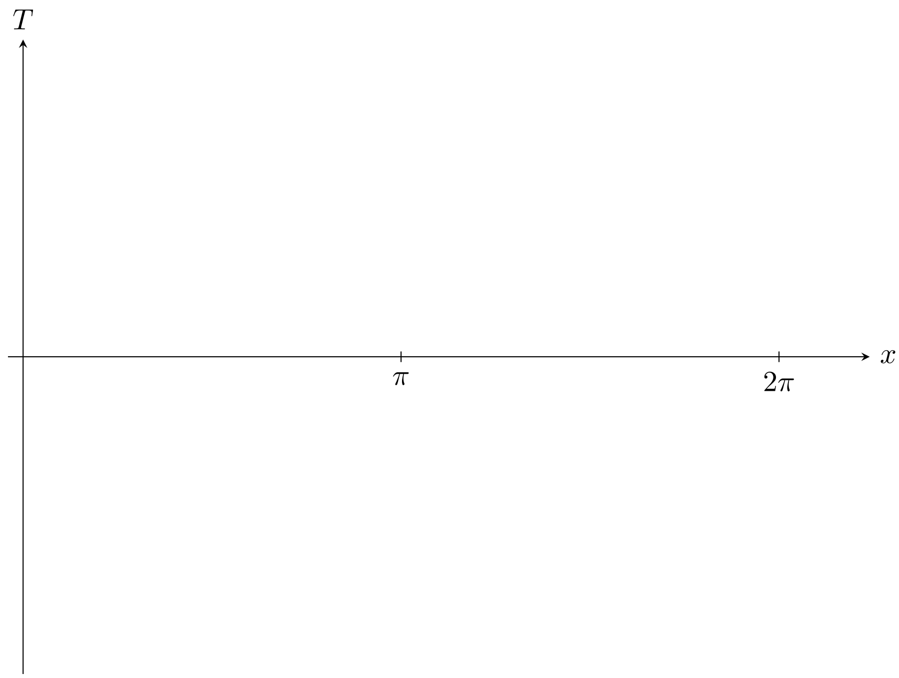
(a) (5 points) Find the steady-state temperature distribution $T(x, y)$ of the infinite rectangular plate whose sides are held at the temperatures described above.

Solution: The key thing to avoid being confused about is that, because $k_n = \frac{n}{2}$, the terms of the series correspond to $n = 2, 6, 10, 14, \ldots$. Alternatively, we can notice that in the basic solutions, the coefficient of y is the negative of the coefficient of x. However we go about it, the steady-state temperature distribution is

$$\begin{aligned} T(x, y) &= \frac{400}{\pi}\left(\sin x e^{-y} + \frac{1}{3}\sin(3x)e^{-3y} + \frac{1}{5}\sin(5x)e^{-5y} + \cdots\right) \\ &= \frac{400}{\pi}\left(T_2 + \frac{1}{3}T_6 + \frac{1}{5}T_{10} + \cdots\right). \end{aligned}$$

(b) (5 points) Alice pokes the plate at $y = 2$ at various values of x. Make a rough plot of what temperatures she finds. That is, plot $T(x, 2)$.

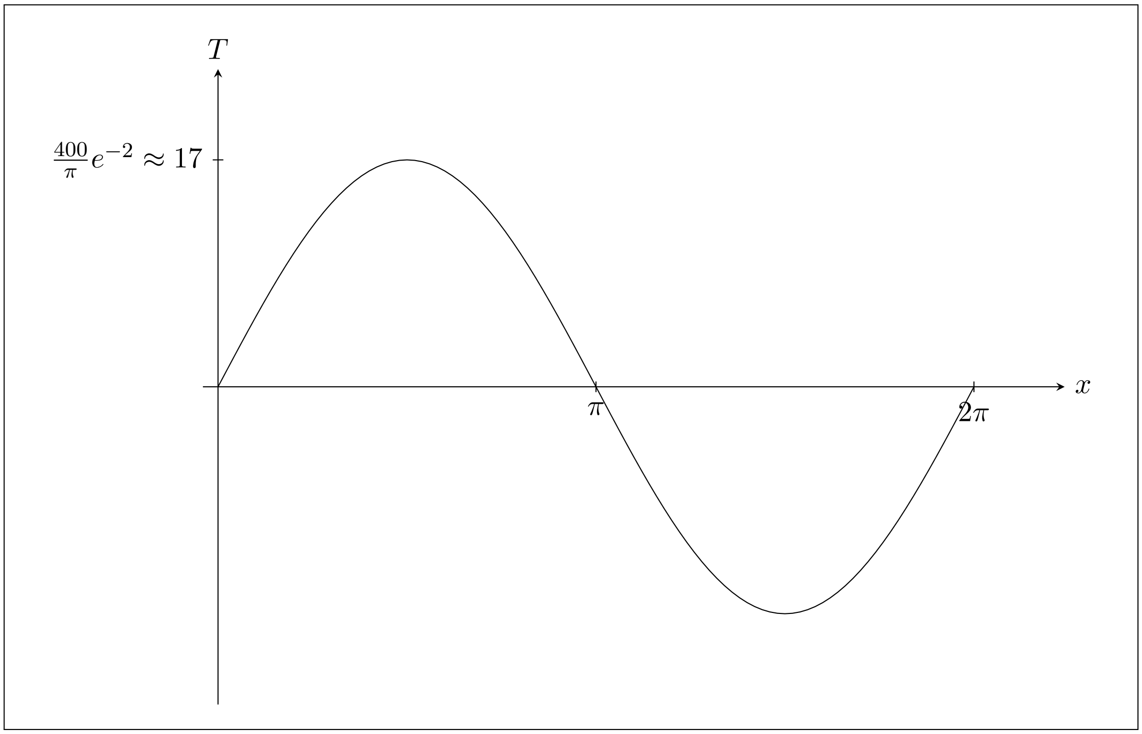
- Hint: At $y = 2$, some terms of your answer to part (a) are much smaller than other terms, so they won't make a difference in a rough sketch or in most real-life applications. You can use rough approximations like $e \approx 2$ or $e \approx 3$ to get a rough sense of which terms matter and which terms don't.
- Label the T-axis with an approximation of the maximum value of your plot. You can leave it in terms of π or e, though making a numerical estimate is a good sanity check to make sure you're on track.



Solution: We look at the amplitudes of the terms at $y = 2$. We note that e^{-2} is somewhere between $\frac{1}{9}$ and $\frac{1}{4}$. Meanwhile, $\frac{1}{3}e^{-3\cdot 2}$ is somewhere between $\frac{1}{3} \cdot \frac{1}{3^6}$ and $\frac{1}{3} \cdot \frac{1}{2^6}$. You don't need to compute these to see that these are much smaller quantities. (The exact quantity $\frac{1}{3}e^{-6}$ is about 164 times smaller than e^{-2}.) Thus, the second term is so much smaller than the first term that you can't even see it looking at a graph, and of course the third term is even smaller. Thus, at $y = 2$, only the first term makes a difference, so

$$T(x, 2) \approx \frac{400}{\pi} \sin x e^{-2}.$$

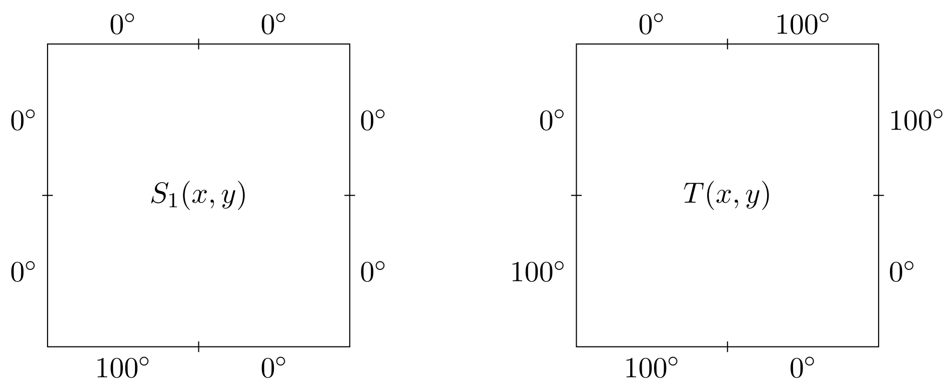
Thus, we just need to draw the graph of $\sin x$, and then mark the T-axis with its amplitude, $\frac{400}{\pi}e^{-2}$. Estimating $\pi \approx e \approx 3$, we see that $\frac{400}{\pi}e^{-2} \approx \frac{400}{27} \approx \frac{400}{25} = 16$. Indeed, the true value is about 17.2, which is good news for Alice, since she's not going to burn her finger, though she might find the -17.2 part of the curve somewhat unpleasant.


T
$\frac{400}{\pi}e^{-2} \approx 17$
x
π
2π

7. (5 points) Consider a square plate of length and width π, with coordinates $0 \le x \le \pi$ and $0 \le y \le \pi$. When half of the bottom side is held at 100 and all other parts of the boundary are held at 0, the steady-state temperature distribution is

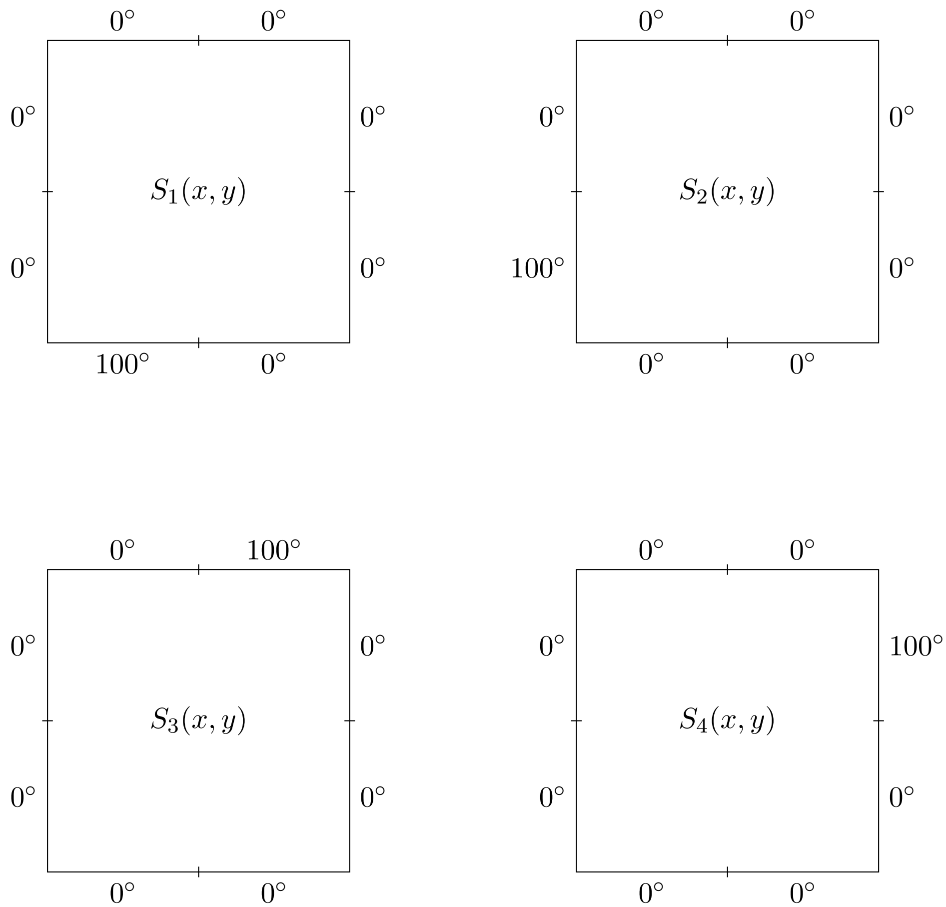
$$S_1(x,y) = \tfrac{200}{\pi}\big(\tfrac{1}{\sinh\pi}\sin x\sinh(\pi-y) + \tfrac{2}{2\sinh 2\pi}\sin 2x\sinh 2(\pi-y) + \tfrac{1}{3\sinh 3\pi}\sin 3x\sinh 3(\pi-y) + \tfrac{1}{5\sinh 5\pi}\sin 5x\sinh 5(\pi-y) + \tfrac{2}{6\sinh 6\pi}\sin 6x\sinh 6(\pi-y) + \tfrac{1}{7\sinh 7\pi}\sin 7x\sinh 7(\pi-y) + \cdots\big)$$

What is the steady-state temperature distribution $T(x,y)$ if half of each side is held at 100 as shown in the diagram below?



Tip: Save time by defining new variables in terms of S_1 rather than copying the long expression, and write your answer in terms of these variables. An acceptable answer could look something like: "Let $S_2(x,y) = S_1(\sqrt{x},\sqrt{y})$. The steady-state temperature distribution is $T(x,y) = S_1(x,y)S_2(x,y)$."

Solution: Our solution method is to consider each side separately and then add the solutions together:



The answer example suggests that we should write S_2, S_3, and S_4 in terms of S_1. We see that the situation in S_2 is the same as the situation in S_1, except with x and y swapped, so

$$S_2(x, y) = S_1(y, x).$$

With S_3, the situation is slightly more complicated, but we see that we can get there from S_1 by reflecting the square vertically and horizontally, which means sending x to $\pi - x$ and y to $\pi - y$. Thus,

$$S_3(x, y) = S_1(\pi - x, \pi - y).$$

Finally, the situation for S_4 is the same as the situation for S_3 but with x and y swapped, so

$$S_4(x, y) = S_3(y, x) = S_1(\pi - y, \pi - x).$$

If we add these four solutions, we'll have the desired boundary value, so the temperature distribution of the square plate is

$$T(x, y) = S_1(x, y) + S_2(x, y) + S_3(x, y) + S_4(x, y).$$

8. (15 points) Consider a metal rod of length $\frac{\pi}{2}$ satisfying the heat equation $\frac{\partial u}{\partial t} = \alpha^2 \Delta u$. In the following two situations, find the "basic answers" and "basic questions." That is, find all solutions u_n of the form $u_n(x,t) = X(x)T(t)$ (the "basic answers") and then, for each u_n, write down the initial heat distribution at $t = 0$ (the "basic question").

Reciting these from memory will help keep you on track but is not sufficient. As in class and in the textbook, start from the form $u = X(x)T(t)$ and use the heat equation to determine what X and T can be.

(a) Find the "basic answers" and "basic questions" if both ends of the rod are in ice at zero degrees. Find all of them in terms of n, and then explicitly write down the first three.

Solution: Plugging $u = XT$ into the heat equation gives

$$\begin{aligned} \frac{\partial u}{\partial t} &= \alpha^2 \Delta u, \\ X\dot{T} &= \alpha^2 X'' T, \\ \frac{\dot{T}}{\alpha^2 T} &= \frac{X''}{X}. \end{aligned}$$

Since the left-hand side does not depend on x and the right-hand side does not depend on t, this expression is a constant. Thus,

$$\begin{aligned} \frac{\dot{T}}{\alpha^2 T} &= C, \\ \dot{T} &= C\alpha^2 T, \\ T &= e^{C\alpha^2 t}, \end{aligned}$$

up to a constant factor.

It's unrealistic for a solution to the heat equation to become hotter and hotter, so C can't be positive, and so we can set $C = -k^2$, so our equation for T becomes

$$T = e^{-(k\alpha)^2 t}$$

Meanwhile, we see that

$$\begin{aligned} \frac{X''}{X} &= C = -k^2, \\ X'' &= -k^2 X, \\ X &= \text{sinusoid}(kx), \end{aligned}$$

for the same constants C and k as above.

Next, the boundary conditions tells us that $X(0) = 0$ and $X(\frac{\pi}{2}) = 0$. The condition $X(0) = 0$ tells us that, up to a constant factor,

$$X = \sin(kx).$$

The second condition tells us that

$$0 = X(\tfrac{\pi}{2}) = \sin(k\tfrac{\pi}{2}).$$

Thus, $k\frac{\pi}{2}$ must be an integer multiple of π. Let $k_n = n\pi$. We see that

$$k_n\frac{\pi}{2} = n\pi$$
$$k_n = 2n.$$

Thus, we have a basic answer

$$u_n = \sin(2nx)e^{-(2n\alpha)^2 t}$$

for any positive integer n. In other words, our basic answers are

$$\sin(2x)e^{-4\alpha^2 t}, \sin(4x)e^{-16\alpha^2 t}, \sin(6x)e^{-36\alpha^2 t}, \ldots .$$

To find corresponding basic questions, we plug in $t = 0$ to find that

$$u_n(0) = \sin(2nx).$$

In other words, our basic questions are

$$\sin 2x, \sin 4x, \sin 6x, \ldots .$$

(b) Find the "basic answers" and "basic questions" if one end is in ice and the other end is insulated. Which end is which is up to whatever you find to be most convenient, but clearly state it. Feel free to refer to work from the previous part; there's no need to write down the same thing twice.

Solution: We can reuse most of our work if the end at $x = 0$ is in ice and the end at $x = \frac{\pi}{2}$ is insulated. The place where things become different is that instead of setting $0 = X(\frac{\pi}{2})$ we should set $0 = X'(\frac{\pi}{2})$. We compute that

$$X = \sin(kx),$$
$$X' = k\cos(kx),$$
$$0 = X'(\tfrac{\pi}{2}) = k\cos(k\tfrac{\pi}{2}).$$

Looking at where cosine is zero, we see that $k\frac{\pi}{2}$ must be an odd multiple of $\frac{\pi}{2}$. There are two clear ways of writing this fact. One is to just explicitly say that $k_n = n$, where n is an odd positive integer. The other is to say that $k_m = 2m - 1$, where m is a positive integer. We'll use the first way. Thus, our basic answers are

$$u_n = \sin(nx)e^{-(n\alpha)^2 t},$$

where n is an odd positive integer. In other words, our basic answers are

$$\sin(x)e^{-\alpha^2 t}, \sin(3x)e^{-9\alpha^2 t}, \sin(5x)e^{-25\alpha^2 t}, \ldots .$$

To find the corresponding basic questions, we plug in $t = 0$ to find that

$$u_n(0) = \sin(nx),$$

where n is an odd positive integer. In other words, our basic questions are

$$\sin x, \sin 3x, \sin 5x, \dots .$$

9. (10 points) A beef patty of thickness 2 cm is taken from a fridge at 0° Celsius and then placed on a stovetop burner. (Real life fridges are usually at around 4° Celsius to avoid freezing the food, but we'll use 0° for simplicity.)

We'll make a reasonable simplifying assumption that the horizontal dimensions of the patty are irrelevant. For the purposes of this problem, we'll simplify the situation a bit further and assume that the temperature of the pan, and hence the temperature of the bottom of the patty, remains at a fixed temperature of 300° Celsius. Again, simplifying things a bit, we'll assume that the top of the patty is warmed by the air and cooled by the evaporation of the water in the patty and thus remains fixed at a temperature of 100° Celsius.

Ground beef has a safe cooking temperature of 70° Celsius, so we'd like to know how long we need to wait until all points of the patty from the bottom to the top are at least at that temperature. The thermal diffusivity of ground beef is approximately $\alpha^2 = 1.5 \times 10^{-3}\,\mathrm{cm}^2/\mathrm{s}$. I recommend just writing α rather than a number wherever it comes up.

- Find a formula for the temperature of the beef patty at any time at any height above the pan.
- You can imagine that, afterwards, you'll estimate the safe cooking temperature by some means, potentially just by putting the formula into a computer, making plots at various values of t, and seeing when the temperature rises above 70° throughout the entire patty.
- You can skip the part of the problem where you find the coefficients of a sine series. To do so, clearly write an expression that looks like

$$e^x = b_1 \sin \sqrt{17}x + b_2 \sin 2\sqrt{17}x + b_3 \sin 3\sqrt{17}x + \cdots .$$

Then, write the rest of your solution in terms of $b_1, b_2, b_3, \ldots$.

Solution: In this case, $l = 2$. We can either use x for the vertical dimension to keep our formulas familiar, or use a variable more commonly associated with the vertical dimension like z. I'll stick with x. At $x = 0$, the temperature is 300. At $x = 2$, the temperature is 100. The steady state temperature u_f is the line joining these two points, so with some algebra we can find that

$$u_f = 300 - 100x.$$

Meanwhile, our initial temperature is $u(0) = 0$. The method for solving such an inhomogeneous problem is to write everything "relative" to the steady state solution and then apply our usual methods. In this case,

$$u(0) - u_f = -(300 - 100x).$$

We expand this expression in a sine series. Since $l = 2$, the sine functions we use are $\sin n\frac{\pi}{2}x$, corresponding to the "basic answers" $u_n = \sin(n\frac{\pi}{2}x)e^{-\left(n\frac{\pi}{2}\alpha\right)^2 t}$. Thus, we write

$$u(0) - u_f = -(300 - 100x) = b_1 \sin \tfrac{\pi}{2}x + b_2 \sin 2\tfrac{\pi}{2}x + b_3 \sin 3\tfrac{\pi}{2}x + \cdots .$$

The corresponding solution is

$$u - u_f = b_1 \sin(\tfrac{\pi}{2}x)e^{-\left(\frac{\pi}{2}\alpha\right)^2 t} + b_2 \sin(2\tfrac{\pi}{2}x)e^{-\left(2\frac{\pi}{2}\alpha\right)^2 t} + b_3 \sin(3\tfrac{\pi}{2}x)e^{-\left(3\frac{\pi}{2}\alpha\right)^2 t} + \cdots$$

This gives us a final answer of

$$\begin{aligned} u &= u_f + b_1 \sin(\tfrac{\pi}{2}x)e^{-\left(\frac{\pi}{2}\alpha\right)^2 t} + b_2 \sin(2\tfrac{\pi}{2}x)e^{-\left(2\frac{\pi}{2}\alpha\right)^2 t} + b_3 \sin(3\tfrac{\pi}{2}x)e^{-\left(3\frac{\pi}{2}\alpha\right)^2 t} + \cdots \\ &= 300 - 100x + b_1 \sin(\tfrac{\pi}{2}x)e^{-\left(\frac{\pi}{2}\alpha\right)^2 t} + b_2 \sin(2\tfrac{\pi}{2}x)e^{-\left(2\frac{\pi}{2}\alpha\right)^2 t} + b_3 \sin(3\tfrac{\pi}{2}x)e^{-\left(3\frac{\pi}{2}\alpha\right)^2 t} + \cdots \end{aligned}$$

It's fine to just write the answer as

$$u = u_f + b_1 u_1 + b_2 u_2 + b_3 u_3 + \cdots ,$$

since u_f and u_n have been defined above.

10. A string of length π and wave velocity v is pulled at its midpoint a distance h, forming a triangle shape.

Let $f(x)$ be the function graphed above for $0 \le x \le \pi$. In a homework problem, you found that the sine series of this function for $0 \le x \le \pi$ is

$$f(x) = \frac{8h}{\pi^2}\left(\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x \pm \cdots\right).$$

(a) (5 points) Find the displacement $u(x,t)$ of a point x along the string at time t, assuming the string has an initial displacement of $f(x)$ and no initial velocity.

Solution: The basic answers for the "guitar problem" are

$$u_n = \sin(nx)\cos(nvt),$$

with corresponding basic questions

$$u_n(0) = \sin(nx).$$

Using our sine series, we see that

$$u(0) = f = \frac{8h}{\pi^2}\left(u_1(0) - \frac{1}{9}u_3(0) + \frac{1}{25}u_5(0) \pm \cdots\right).$$

Hence,

$$u = \frac{8h}{\pi^2}\left(u_1 - \frac{1}{9}u_3 + \frac{1}{25}u_5 \pm \cdots\right),$$

where the u_n are as above.

(b) (5 points) Find the displacement $u(x,t)$ of a point x along the string at time t if, instead, the string is given an initial velocity of $f(x)$, and the string has no initial displacement.

Solution: The basic answers for the "piano problem" are

$$u_n = \sin(nx)\sin(nvt).$$

Our basic questions are now initial velocities, so we compute that

$$\dot{u}_n = \sin(nx)(nv)\cos(nvt).$$

Thus, our basic questions, that is, the corresponding initial velocities, are

$$\dot{u}_n(0) = (nv)\sin(nx)$$

We then compute that

$$\begin{aligned}\dot{u}(0) = f &= \frac{8h}{\pi^2}\left(\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x \pm \cdots\right) \\ &= \frac{8h}{\pi^2}\left(\frac{\dot{u}_1(0)}{v} - \frac{1}{9}\frac{\dot{u}_3(0)}{3v} + \frac{1}{25}\frac{\dot{u}_5(0)}{5v} \pm \cdots\right) \\ &= \frac{8}{\pi^2}\frac{h}{v}\left(\dot{u}_1(0) - \frac{1}{27}\dot{u}_3(0) + \frac{1}{5^3}\dot{u}_5(0) \pm \cdots\right).\end{aligned}$$

Thus, our solution is

$$u = \frac{8}{\pi^2} \frac{h}{v} \left(u_1 - \frac{1}{27} u_3 + \frac{1}{5^3} u_5 \pm \cdots \right),$$

where the u_n are as above.

Question	Points	Score
1	5	
2	10	
3	5	
4	25	
5	5	
6	10	
7	5	
8	15	
9	10	
10	10	
Total:	100	