Math 308 Final

May 3, 2018

Name: ____

- Show your work. If you solve a problem with anything other than a straightforward computation, write one complete sentence explaining what you're doing.
  - For example, if you're computing a cross product using the standard method, just show your computation.
  - But if, for example, you find that a line integral is zero without actually computing the line integral, you need to write one complete sentence convincing an imaginary peer that that's true.
- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
  - Make a note on the printed page that your work continues on the back of the previous page.
  - On the back of the previous page, put a box around the work that you want graded.
- There are 10 questions, worth between 5 and 25 points each.
  - The problems are ordered by topic, not by difficulty.
- Guidelines for acceptable answers for Fourier series questions by way of analogy:
  - Acceptable: $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.
  - Acceptable: $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$. Include enough terms so the pattern is clear.
  - Unacceptable: $c_n = \frac{1}{n!}$.
  - Acceptable: $e^x = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$, where $c_n = \frac{1}{n!}$.
  - Acceptable: $e^x = \sum_{n=0}^{\infty} c_n x^n$, where $c_n = \frac{1}{n!}$.

1. (5 points) According to my phone, I am currently at a latitude of $38.6487°N$ and a longtitude of $90.3057°W$ at an altitude of about 500 feet above sea level. Using an online tool to convert this position to rectangular coordinates while taking into account the fact that the Earth is not perfectly spherical, my position vector is approximately

$$-(5000\,\mathrm{km})\mathbf{j} + (4000\,\mathrm{km})\mathbf{k}.$$

What is my velocity vector with respect to the center of the Earth? Include units.

Facts, some of which are helpful:

- The standard rectangular coordinates are centered on the center of the Earth, with the z-direction pointing along the Earth's axis of rotation towards the North pole, and with Greenwich, England in the xz-plane with a positive x-value.
- The Earth takes about 23 hours and 56 minutes to complete a full rotation, going counterclockwise from the persepctive of someone looking down on the North pole. (The extra 4 minutes of the day come from the motion of the Earth around the Sun.) For the purposes of this problem, approximate this rotation rate as 0.3 radians/hour.

2. Consider the triangle in the xy-plane with vertices $(0,0)$, $(6,0)$, and $(0,6)$. Consider the vector field

$$\mathbf{F} = -y\mathbf{i} + x\mathbf{j}.$$

Let γ be the closed path going around the triangle counter-clockwise.

(a) (5 points) Compute $\oint_\gamma \mathbf{F} \cdot d\mathbf{r}$ directly, without using Stokes' (Green's) Theorem.

(b) (5 points) Compute $\oint_\gamma \mathbf{F} \cdot d\mathbf{r}$ using Stokes' (Green's) Theorem.

3. (5 points) Consider the three vector fields below.

$$\begin{aligned}\mathbf{V}_1 &= (2x+y+z)\mathbf{i} + (x+y+2z)\mathbf{j} + (x-2y-z)\mathbf{k}.\\ \mathbf{V}_2 &= (x-2y-z)\mathbf{i} + (2x+y+z)\mathbf{j} + (x+y+2z)\mathbf{k}.\\ \mathbf{V}_3 &= (x+y+2z)\mathbf{i} + (x-2y-z)\mathbf{j} + (2x+y+z)\mathbf{k}.\end{aligned}$$

Which of these three vector fields has a vector potential? That is, which of these three vector fields is equal to the curl of another vector field?

For a complete solution, you should explain

- why one of the vector fields has a vector potential, and
- why the two other vector fields do not have a vector potential.

Finding a vector potential certainly counts as a valid explanation for the first bullet point, but doing so is not required.

4. Consider the 2π-periodic function defined by

$$f(x) = \begin{cases} 7 & \text{if } 0 < x < \pi \\ 0 & \text{if } \pi < x < 2\pi. \end{cases}$$

Be sure to follow the guidelines for acceptable answers on the exam cover.

(a) (10 points) Find the Fourier series of f in terms of sines and cosines. Compute directly, without using your answer to part (b).

(b) (5 points) Find the Fourier series of f in terms of complex exponentials. Compute directly, without using your answer to part (a).

(c) (5 points) Show that your answers in parts (a) and (b) are equal.

(d) (5 points) Apply Parseval's Theorem to either the series in part (a) or part (b). Doing both is not required but will let you check your work. Your final answer should be an equation with a number on one side and a series on the other side.

5. (5 points) Consider the function f on the whole real line defined by

$$f(x) = \begin{cases} 17 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Compute the Fourier transform of f. The guidelines for acceptable answers for Fourier series apply here as well, except you'll have an integral instead of a sum.

6. Consider an infinite rectangular plate of width 2π, so $0 \leq x \leq 2\pi$ and $0 \leq y < \infty$. The left and right sides of the rectangular plate are held at 0 degrees. The bottom side is held at $f(x)$, where

$$f(x) = \begin{cases} 100 & \text{if } 0 < x < \pi, \\ -100 & \text{if } \pi < x < 2\pi. \end{cases}$$

Helpful facts:

- The following functions T_n are solutions to the equation $\Delta T_n = 0$.

$$T_n(x, y) = \sin(k_n x)e^{-k_n y},$$

where $k_n = \frac{n}{2}$, and n is a positive integer.

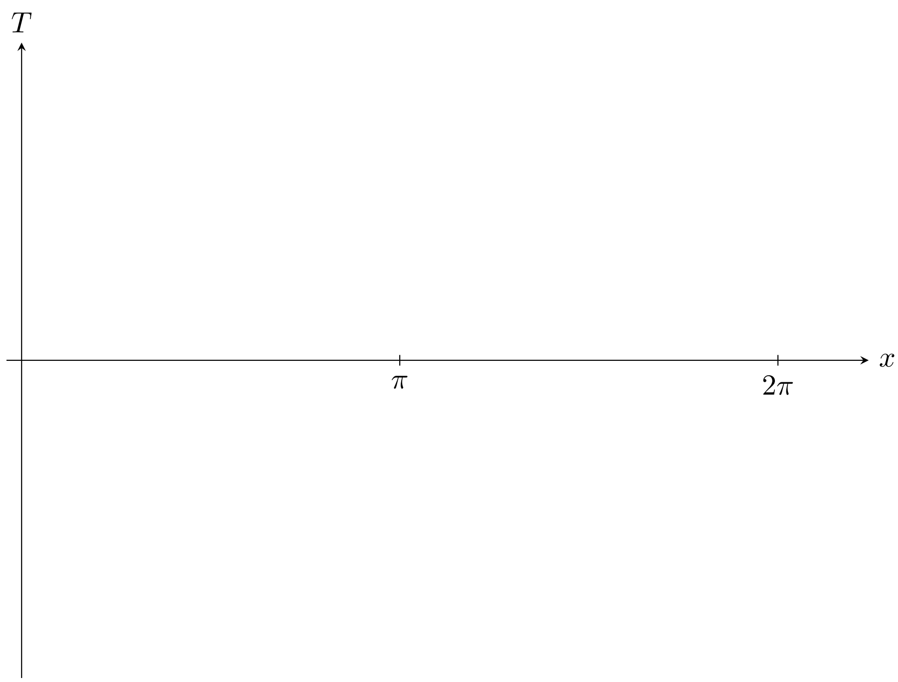
- The sine series for $f(x)$ is

$$f(x) = \frac{400}{\pi}\left(\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \cdots\right)$$

(a) (5 points) Find the steady-state temperature distribution $T(x, y)$ of the infinite rectangular plate whose sides are held at the temperatures described above.

(b) (5 points) Alice pokes the plate at $y = 2$ at various values of x. Make a rough plot of what temperatures she finds. That is, plot $T(x, 2)$.

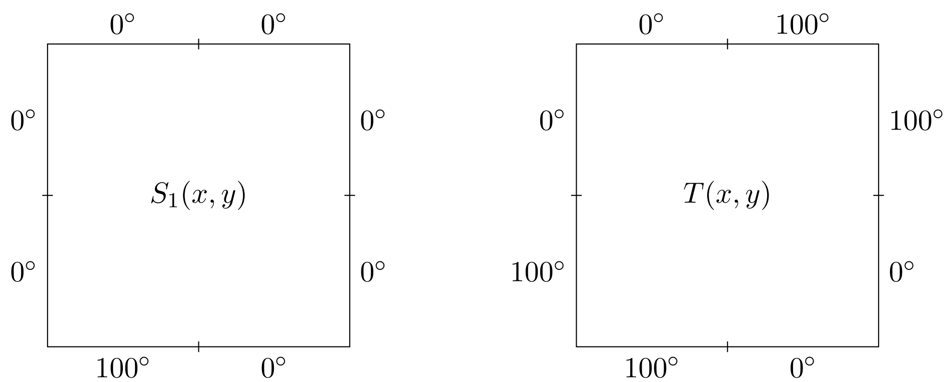
- Hint: At $y = 2$, some terms of your answer to part (a) are much smaller than other terms, so they won't make a difference in a rough sketch or in most real-life applications. You can use rough approximations like $e \approx 2$ or $e \approx 3$ to get a rough sense of which terms matter and which terms don't.
- Label the T-axis with an approximation of the maximum value of your plot. You can leave it in terms of π or e, though making a numerical estimate is a good sanity check to make sure you're on track.



7. (5 points) Consider a square plate of length and width π, with coordinates $0 \le x \le \pi$ and $0 \le y \le \pi$. When half of the bottom side is held at 100 and all other parts of the boundary are held at 0, the steady-state temperature distribution is

$$S_1(x,y) = \frac{200}{\pi}\Big(\frac{1}{\sinh \pi}\sin x \sinh(\pi - y) + \frac{2}{2\sinh 2\pi}\sin 2x \sinh 2(\pi - y) + \frac{1}{3\sinh 3\pi}\sin 3x \sinh 3(\pi - y) + \frac{1}{5\sinh 5\pi}\sin 5x \sinh 5(\pi - y) + \frac{2}{6\sinh 6\pi}\sin 6x \sinh 6(\pi - y) + \frac{1}{7\sinh 7\pi}\sin 7x \sinh 7(\pi - y) + \cdots\Big)$$

What is the steady-state temperature distribution $T(x,y)$ if half of each side is held at 100 as shown in the diagram below?



Tip: Save time by defining new variables in terms of S_1 rather than copying the long expression, and write your answer in terms of these variables. An acceptable answer could look something like: "Let $S_2(x,y) = S_1(\sqrt{x}, \sqrt{y})$. The steady-state temperature distribution is $T(x,y) = S_1(x,y)S_2(x,y)$."

8. (15 points) Consider a metal rod of length $\frac{\pi}{2}$ satisfying the heat equation $\frac{\partial u}{\partial t} = \alpha^2 \Delta u$. In the following two situations, find the "basic answers" and "basic questions." That is, find all solutions u_n of the form $u_n(x,t) = X(x)T(t)$ (the "basic answers") and then, for each u_n, write down the initial heat distribution at $t = 0$ (the "basic question").

   Reciting these from memory will help keep you on track but is not sufficient. As in class and in the textbook, start from the form $u = X(x)T(t)$ and use the heat equation to determine what X and T can be.

   (a) Find the "basic answers" and "basic questions" if both ends of the rod are in ice at zero degrees. Find all of them in terms of n, and then explicitly write down the first three.

   (b) Find the "basic answers" and "basic questions" if one end is in ice and the other end is insulated. Which end is which is up to whatever you find to be most convenient, but clearly state it. Feel free to refer to work from the previous part; there's no need to write down the same thing twice.

9. (10 points) A beef patty of thickness 2 cm is taken from a fridge at 0° Celsius and then placed on a stovetop burner. (Real life fridges are usually at around 4° Celsius to avoid freezing the food, but we'll use 0° for simplicity.)

   We'll make a reasonable simplifying assumption that the horizontal dimensions of the patty are irrelevant. For the purposes of this problem, we'll simplify the situation a bit further and assume that the temperature of the pan, and hence the temperature of the bottom of the patty, remains at a fixed temperature of 300° Celsius. Again, simplifying things a bit, we'll assume that the top of the patty is warmed by the air and cooled by the evaporation of the water in the patty and thus remains fixed at a temperature of 100° Celsius.

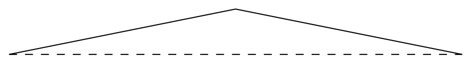
   Ground beef has a safe cooking temperature of 70° Celsius, so we'd like to know how long we need to wait until all points of the patty from the bottom to the top are at least at that temperature. The thermal diffusivity of ground beef is approximately $\alpha^2 = 1.5 \times 10^{-3}\,\text{cm}^2/\text{s}$. I recommend just writing α rather than a number wherever it comes up.

   - Find a formula for the temperature of the beef patty at any time at any height above the pan.
   - You can imagine that, afterwards, you'll estimate the safe cooking temperature by some means, potentially just by putting the formula into a computer, making plots at various values of t, and seeing when the temperature rises above 70° throughout the entire patty.
   - You can skip the part of the problem where you find the coefficients of a sine series. To do so, clearly write an expression that looks like

     $$e^x = b_1 \sin \sqrt{17}x + b_2 \sin 2\sqrt{17}x + b_3 \sin 3\sqrt{17}x + \cdots .$$

     Then, write the rest of your solution in terms of $b_1, b_2, b_3, \ldots$.

10. A string of length π and wave velocity v is pulled at its midpoint a distance h, forming a triangle shape.



Let $f(x)$ be the function graphed above for $0 \le x \le \pi$. In a homework problem, you found that the sine series of this function for $0 \le x \le \pi$ is

$$f(x) = \frac{8h}{\pi^2}\left(\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x \pm \cdots\right).$$

(a) (5 points) Find the displacement $u(x, t)$ of a point x along the string at time t, assuming the string has an initial displacement of $f(x)$ and no initial velocity.

(b) (5 points) Find the displacement $u(x, t)$ of a point x along the string at time t if, instead, the string is given an initial velocity of $f(x)$, and the string has no initial displacement.

Question	Points	Score
1	5	
2	10	
3	5	
4	25	
5	5	
6	10	
7	5	
8	15	
9	10	
10	10	
Total:	100	