MATH 308 COURSE SUMMARY

Approximately a third of the exam cover the material from the first two midterms, that is, chapter 6 and the first six sections of chapter 7. The rest of the exam will cover the remaining material, that is, the rest of chapter 7 and the first four sections of chapter 13. The textbook has examples; making sure you can do them with the book closed is a good starting point for studying, followed by doing additional problems in the areas that are giving you the most trouble.

Chapter 6

Section 1. The short introduction is good to read.

Section 2. It's all well and good to be able to compute dot products and cross products, but what are they actually good for? This section covers some basic questions that these tools answer.

Section 3. The triple scalar product lets us compute volumes. There's a formula that makes it easier to compute triple vector products that takes advantage of the fact that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is in the plane spanned by $\mathbf{B}$ and $\mathbf{C}$.

Section 4. If a particle moves around, its position vector depends on time. We can take the derivative of the position vector to find the particle's velocity vector. The product rule works as expected for dot products and cross products. Taking derivatives is straightforward in rectangular coordinates; in polar coordinates we need to remember to take derivatives of the basis vectors, too.

Section 5. A scalar field is a fancy word for a real-valued function, thought of as having a number, such as temperature, assigned to every point in space. Similarly, a vector field assigns a vector, such as wind direction, to every point in space.

Section 6. You can compute the gradient of a scalar field to get a vector field. The gradient tells you the directional derivative of the scalar field. If you have a surface defined by an equation, the gradient points in the normal direction. On a map, this fact can be reinterpreted as saying that the gradient is perpendicular to the contour lines. The gradient can be written in cylindrical or spherical coordinates; if these formulas are provided to you, you should be able to apply them.

Section 7. You can compute the curl of a vector field to get a vector field, and you can compute the divergence of a vector field to get a scalar field. You can combine the gradient, curl, and divergence to get other operators, such as the Laplacian. There are various vector identities and heuristics for coming up with them. There are fairly complicated expressions for the divergence and Laplacian in cylindrical and spherical coordinates.

Section 8. You can compute line integrals of vector fields along curves. You need to parametrize the curve to do so, but the answer doesn't depend on the parametrization. Line integrals have a physical interpretation as work done. For a conservative vector field, line integrals don't depend on the path, you can find a scalar potential representing potential energy that lets you evaluate line integrals just by evaluating the scalar potential at the endpoints. If a vector field is conservative, its curl is zero. Assuming the domain has no holes, the reverse is true: if the curl a vector field is zero, then it is conservative. But if the domain has a hole you might have a vector field that has zero curl but is not conservative; you can't find a vector potential. It's helpful to convert a vector field $\mathbf{F}$ to a differential $\mathbf{F} \cdot d\mathbf{r}$.

Section 9. We warm up for the three-dimensional Stokes' and divergence theorems with their two dimensional versions.

Section 10. The divergence theorem relates a volume integral of a divergence to a surface integral. Interpreting the divergence as creation or expansion, you can interpret the volume integral as total net creation and expansion, and the surface integral as how much stuff flows out through the boundary. In other words, you can tell if stuff is being created or expanded inside the volume by seeing if it's leaking out. Gauss's law gives an electromagnetic interpretation, where charges are sources causing electric displacement $\mathbf{D}$.

Section 11. Stokes' theorem relates a surface integral of a curl with a line integral on the boundary. The orientation of the surface matters, and given an orientation of the surface, there is a compatible orientation of its boundary. Interpreting the curl as rotation and drawing pictures helps get orientations right. If the water inside a pool is rotating, it will push someone around the boundary of the pool. There's also an electromagnetic interpretation. The section revisits conservative fields more carefully. Next, note that if a vector field is the curl of another vector field, its divergence is zero. Assuming there are no holes in the domain, the reverse is true: if the divergence of a vector field is zero, you can find a vector potential. For the scalar potential, you could add a constant to get another valid scalar potential. For the vector potential, there are many more choices: Given a vector potential, you can add the gradient of any function to it to get another valid vector potential. Assuming there are no holes in the domain, those are all the possibilities. That is, if you find two vector potentials for the same vector field, their difference will be a gradient. With a short computation, you can see that this fact is a consequence of the fact that if the curl of a vector field is zero, then it is a gradient.

Chapter 7

Section 1. The short introduction is good to read.

Section 2. There are various pieces of terminology dealing with sinusoids: amplitude, period, frequency, wavelength, etc.

Section 3. This section provides good intuition for Fourier series and Fourier transforms in terms of music, and discusses other situations where these are valuable tools.

Section 4. This section discusses average value, which you are already familiar with. However, there are many situations, including sinusoids, squares of sinsuoids, complex exponentials, odd functions, etc., where you can compute the average value without computing an antiderivative. Doing so is important both to save time and to get a better sense of what's going on.

Section 5. You can compute Fourier series. Understanding the formulas helps with avoiding mistakes when remembering them. To get a sense of why the formula is what it is, try a simple function like $23\cos(17x)$. What should the Fourier coefficients be? Does that match with what your formula gives?

Section 6. As you know from the radius of convergence of power series, a power series will converge for some values of x and not converge for other values of x. Even if it converges, it might not converge to the value of the original function, as in the case of e^{-1/x^2}. Fourier series are much nicer. For any reasonable function that you are likely to encounter, the Fourier series will converge at every value of x, and its sum will be the value of the original function if it is continuous at x and the midpoint of the jump if it is not. The delta function is not "reasonable" in the sense; Fourier series of things like the delta function don't converge, though they are still very useful both in theory and in practice, but it's best to learn tools beyond the scope of this class before working with them.

Section 7. You can write Fourier series using complex exponentials; the formulas become simpler. The two formulations are equivalent; you can convert a Fourier series in terms of complex exponentials into a Fourier series in terms of sines and cosines, and vice versa.

Section 8. You can write Fourier series on intervals of length $2l$ rather than 2π. This causes a lot of $\frac{\pi}{l}$ factors to appear in all the formulas. Again, understanding the formulas helps with avoiding mistakes when remembering them. Try a $2l$-periodic function like $23\cos(17\frac{\pi}{l}x)$. What should the Fourier coefficients be? Does that match with what your formula gives?

Section 9. The Fourier series of an even function will only have cosine terms, and the Fourier series of an odd function will only have sine terms. Given a function on an interval of length l, we can extend it to an even function on an interval of length $2l$, or to an odd function of length $2l$. As a result, we can write any function as a sum of sines only, or as a sum of cosines only. The key distinction is that in a Fourier series, all of the sines and consines we used completed full periods on the interval in question. But in a Fourier sine series or Fourier cosine series, we also use functions that only complete half a period on the interval in question. The same trick for remembering the formulas works here; make sure that it works for a basic function like $23\cos(17\frac{\pi}{l}x)$. Somewhat randomly, this section also talks about differentiating Fourier series. At this level, we want to avoid delta functions, so start with a function f that has no jumps, but it's okay if its derivative f' has jumps. The Fourier series of f' will be the derivative of the Fourier series for f. If you already know the Fourier series for f, this fact saves some work.

Section 10. This section discusses music again. It's good for building intuition about Fourier series and about which terms of the series are more important than others.

Section 11. In many contexts, the integral of the square of a function is important and represents something like energy, power, or intensity. Parseval's theorem says that we can find this energy by computing the energy of each term of the Fourier series individually, and then adding up these energies to get the total. In other words, you can get the total energy directly from the coefficients. Most series won't work so nicely, but Fourier series do. In practice, this means that when we talk about something like the power output of the sun, we can reasonably talk about it as the power output in the visible spectrum, plus the power output in the infrared spectrum, plus the power output in the ultraviolet spectrum. Unlike humans or even electrons, light at different frequencies can't combine to become something more powerful than the sum of its parts. Likewise, light at different frequencies can't interfere to become weaker than the sum of its parts either.

Section 12. The ideas behind Fourier series apply to non-periodic functions as well, giving us the Fourier transform. The best way to get the formula straight is to understand the analogy between $g(\alpha)$ and c_n, but always keep in mind that n must be an integer, whereas α could be $\sqrt{2}-e$ for all we know. For odd functions, we have the Fourier sine transforms, and for even functions, we have the Fourier cosine transforms. The book changes conventions here, using factors of $\sqrt{\frac{2}{\pi}}$ out front in equations (12.14) and (12.15). As long as the two factors out front multiply to $\frac{2}{\pi}$, you'll get the right answer, so if you'd like, you can use 1 for the factor in the first equation and $\frac{2}{\pi}$ for the factor in the second equation, putting the formulas more in line with equations (9.4) and (9.5). Or you can keep it as is, enjoying the symmetry between the two formulas. Parseval's theorem works similarly for Fourier transforms to how it works for Fourier transforms.

Chapter 13

Section 1. The introduction helps place these ideas in a larger context of partial differential equations.

Section 2. This section talks about two problems: the infinite rectangular plate and the finite rectangular plate, where the temperature on the bottom edge is given and the temperature on the other sides is zero. Linearity tells us that adding two solutions will also give us a solution, so we proceed by finding basic solutions of the form $X(x)Y(y)$. It's good to remember both what the basic solutions are and how to come up with them; gaps in remembering the formula can be filled by knowing where it came from, and gaps in remembering how to come up with the formula can be filled by knowing what you're aiming for. Each basic solution has a corresponding "question," that is, a value for the temperature at the bottom edge. Given a "question," that is, a temperature of the bottom edge, writing it as a sine series lets you write it as a linear combination of basic questions, and hence lets you write your answer as the corresponding linear combination of basic answers. At the end, the section talks about how to solve the rectangular plate problem when the temperature on the other three sides is given rather than just zero. Linearity saves the day. We can keep one side as is and set the other three sides to zero, and solve the problem. Then we can do the same thing for another side. After we've solved four problems, one for each side, we can add the four answers together to get a solution with the right boundary value on all four sides.

Section 3. This section talks about the heat/diffusion problem. The initial discussion works in three dimensions, but at the end of the day we focus on the one-dimensional problem, where either there is only one-dimension, such as a rod, or where two of the dimensions don't matter, such as a wall. As in the previous section, you can find basic solutions of the form $X(x)T(t)$. The key distinction from before is that you've got a $-k^2$ in the exponential instead of just $-k$ like in the previous problem. There's some variations depending on whether the ends are in ice, so $X = 0$ at the ends, or the ends are insulated, so $X' = 0$ at the ends. You could also have one of each. You can also solve an inhomogeneous problem, where the ends are fixed at arbitrary values, perhaps one at 20 and one at 100. Linearity saves the day here as well. You can find the steady state solution, and subtract it to convert to a problem where the ends are back at zero. Solve this corresponding homogeneous problem as before, and then add the steady state solution back in to get the answer to the original inhomogeneous problem. We haven't talked about the Schrödinger equation, but when you see it in another class, remember the connection.

Section 4. This section talks about the wave equation. Again, the initial discussion works in three dimensions, but eventually we focus on the one-dimensional problem. The key difference here is that the "question" consists of two pieces of data: the initial position and the initial velocity. We separated the discussion into a "guitar" problem where the initial velocity is zero and a "piano" problem where the initial position is zero. But as with the problem of the finite rectangular plate with four given sides, if we're given a question that's a mix of the two with both an initial position and an initial velocity, we can solve the two problems separately and then add the solutions together to get our answer. The constant v in the wave equation is the speed at which waves travel along a long string made of this material. The possible values of k given by our basic solutions have a physical interpretation and at the end of the day tell us the frequencies of sound that this string is able to produce.

After the exam

There are a couple topics beyond the scope of this course that I hinted at throughout. If, after spending a couple weeks relaxing on break, you'd like to know more about them, here's what to look for.

Expressions like $dx \wedge dy$. These are *differential forms*, sometimes disguised as *antisymmetric tensors*. Learning how to integrate differential forms let you use a single set of tools to compute line intergrals, surface integrals, and volume integrals, and lets you understand the fundamental theorem of calculus, Stokes's theorem, and the divergence theorem, as all being a single idea. A class in differential geometry or manifolds is very likely to cover this topic, but so might a course on tensors, depending on the focus.

The delta function. The delta function is a *distribution*, which basically means that you know how to evaluate expressions like $\int f(x)\delta(x)\,dx$ if f is a nice smooth function, but maybe not if f has a jump discontinuity. A course on distribution theory or functional analysis is likely to talk about these, but you'll also see it in physics courses where things might be less precise. You can take the derivative of any distribution, even the delta function. There are cool facts about such derivatives of distributions, such as that if $\int |f'(x)|^2\,dx < \infty$, then f is continuous. But, in two dimensions, if $\int \|\nabla f\|^2\,dx\,dy < \infty$, then f doesn't have to be continuous, but the simplest counterexample has a discontinuity at the origin that looks like $\log(\log(4/r))$. But if we take two derivatives using the Laplacian, if $\int |\Delta f|^2\,dx\,dy < \infty$, then f has to be continuous. The same result also works in dimension three, but not in dimension four, where, again, the simplest counterexample has a discontinuity at the origin that looks like $\log(\log(4/r))$.

The heat and wave equations in higher dimensions and electron orbitals. We talked about a vibrating string, but what about a drum? The book continues with that in section 6 of chapter 13, though reading section 5 first is a good idea. Moving up to three dimensions, the "basic answers" for the heat equation on the sphere, also known as *spherical harmonics*, are in section 7, though there's some background from chapter 12 needed; just follow the references. Spherical harmonics also give the solutions for a particle in a spherical box. Adding in a potential energy term to the differential equation turns the box into a hydrogen atom and turns the "basic answers" into the electron orbitals you might have learned about in chemistry class. A physics class is probably the best place to learn about these.