Math 5051 Midterm

October 19, 2018

Name: ____

- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
  - Make a note on the printed page that your work continues on the back of the previous page.
  - On the back of the previous page, put a box around the work that you want graded.
- Give and use definitions from the book or from class.
- You may use any results you remember from the book or from class as long as they are more basic than the result you're asked to prove.

1. (a) (5 points) Let $\mathcal{M}$ be a σ-algebra on X. State the definition of a *measure* μ on $(X, \mathcal{M})$.

Solution: A measure is a function $\mu\colon \mathcal{M} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E_j)$$

whenever E is the countable disjoint union of sets E_j in $\mathcal{M}$.

(b) (5 points) State and prove the property of a measure called *monotonicity*.

Solution:
Claim. *If $A \subseteq B$ and A and B are in $\mathcal{M}$, then $\mu(A) \leq \mu(B)$.*

Proof. We know that $B \setminus A = B \cap A^c \in \mathcal{M}$, and we know that $B = A \sqcup (B \setminus A)$. Therefore, by additivity,

$$\mu(B) = \mu(A) + \mu(B \setminus A).$$

Because $\mu(B \setminus A) \geq 0$, we conclude that $\mu(B) \geq \mu(A)$. □

(c) (10 points) State and prove the property of a measure called *continuity from below*.

Solution:
Claim. *Let $E_1 \subseteq E_2 \subseteq \cdots$ be an increasing sequence of sets in $\mathcal{M}$, and let $E = \bigcup_{n=1}^{\infty} E_n$. Then*

$$\mu(E) = \lim_{n\to\infty} \mu(E_n).$$

Proof. Let $F_1 = E_1$, $F_2 = E_2 \setminus E_1$, and, in general, $F_n = E_n \setminus E_{n-1}$. Then the F_n are disjoint and

$$\bigsqcup_{j=1}^{\infty} F_j = E, \qquad \bigsqcup_{j=1}^{n} F_j = E_n.$$

Thus, using countable additivity and finite additivity, we have that

$$\mu(E) = \sum_{j=1}^{\infty} \mu(F_j), \qquad \mu(E_n) = \sum_{j=1}^{n} \mu(F_j).$$

By definition, $\sum_{j=1}^{\infty} \mu(F_j) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(F_j)$. Hence $\mu(E) = \lim_{n\to\infty} \mu(E_n)$, as desired. □

2. Let $(X, \mathcal{M})$ be a measurable space.

   (a) (5 points) Let $f\colon X \to \overline{\mathbb{R}}$. Define what it means for f to be *measurable*.

   Solution: The Borel σ-algebra $\mathcal{B}_{\overline{\mathbb{R}}}$ is the σ-algebra on $\overline{\mathbb{R}}$ generated by the open sets in $\mathbb{R}$ and the singleton sets $\{\infty\}$ and $\{-\infty\}$. A function f is measurable if $f^{-1}(B)$ is in $\mathcal{M}$ for any $B \in \mathcal{B}_{\overline{\mathbb{R}}}$.

   (b) (15 points) Let $f_n\colon X \to \overline{\mathbb{R}}$ be a sequence of measurable functions. Show that the function $g\colon X \to \overline{\mathbb{R}}$ defined by $g(x) = \sup_n f_n(x)$ is also measurable.

   Solution: We showed in class that $\mathcal{B}_{\overline{\mathbb{R}}}$ is generated by the rays $(a, \infty]$, so it suffices to show that $g^{-1}((a, \infty])$ is in $\mathcal{M}$ for all a.

   By the definition of supremum, we know that if $g(x) > a$, then there exists an n such that $f_n(x) > a$ as well. Conversely, if there exists an n such that $f_n(x) > a$, then $g(x) \geq f_n(x) > a$ as well. The set $\{x \mid g(x) > a\}$ is also known as $g^{-1}((a, \infty])$, and likewise for $f_n^{-1}((a, \infty])$. We conclude then that

   $$g^{-1}((a, \infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((a, \infty]).$$

   Because f_n is measurable, we know that $f_n^{-1}((a, \infty]) \in \mathcal{M}$. Thus, the countable union of these sets, $g^{-1}((a, \infty])$, is in $\mathcal{M}$ as well.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space, let $f\colon X \to [0, \infty]$ be a measurable function, and assume that $\int f < \infty$.

(a) (10 points) Show that the set $\{x \mid f(x) = \infty\}$ has measure zero.

Solution: Let $E = \{x \mid f(x) = \infty\}$. For any $M \in (0, \infty)$, let $\phi = M\chi_E$. Then ϕ is a simple function, so $\int \phi = M\mu(E)$, and $\phi \leq f$, so $\int \phi \leq \int f$. We conclude that

$$\mu(E) < \frac{\int f}{M}.$$

Because $\int f$ is finite and M can be arbitrarily large, we conclude that $\mu(E) = 0$.

(b) (10 points) Show that the set $\{x \mid f(x) > 0\}$ is σ-finite.

Solution: Let $E = \{x \mid f(x) > 0\}$, and let

$$E_n = \left\{x \mid f(x) > \tfrac{1}{n}\right\}.$$

We see that $E = \bigcup_{n=1}^{\infty} E_n$, since $f(x) > 0$ if and only if $f(x) > \frac{1}{n}$ for some n.
It remains to show that $\mu(E_n) < \infty$ for all n. Given n, let $\phi = \frac{1}{n}\chi(E_n)$. Then ϕ is a simple function, so $\int \phi = \frac{1}{n}\mu(E_n)$. Moreover, $\phi \leq f$. Indeed, for $x \in E_n$, $\phi(x) = \frac{1}{n} < f(x)$, and for $x \notin E_n$, $\phi(x) = 0 \leq f(x)$. Thus, $\int \phi \leq \int f$. Multiplying this equation by n, we conclude that $\mu(E_n) \leq n \int f < \infty$.

4. (15 points) Let $(X, \mathcal{M}, \mu)$ be a measure space, let $f\colon X \to [0, \infty]$ be a measurable function, and assume that $\int_X f < \infty$.

   Show that you can approximate $\int_X f$ by $\int_E f$ where E has finite measure. That is, for every $\epsilon > 0$, show that there exists an $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_X f - \int_E f < \epsilon$.

Solution: By definition, $\int f = \sup\{\int \phi \mid 0 \le \phi \le f \text{ and } \phi \text{ is simple}\}$. Thus, there exists a simple function ϕ such that $0 \le \phi \le f$ and $\int f - \int \phi < \epsilon$.

Let $\phi = \sum_{j=1}^n a_j \chi_{E_j}$. By definition, $\int \phi = \sum_{j=1}^n a_j \mu(E_j)$. We know that $\int \phi \le \int f < \infty$, so $a_j \mu(E_j) < \infty$ for all j, and so either $a_j = 0$ or $\mu(E_j) < \infty$. Let E be the union of the E_j where $a_j > 0$. This is a finite union of finite-measure sets, so $\mu(E) < \infty$.

Moreover, for any $x \notin E$, we know that $\phi(x) = 0$. Thus, $\phi \le f\chi_E$. Indeed, for $x \notin E$, $\phi(x) = 0 = (f\chi_E)(x)$, and for $x \in E$, $\phi(x) \le f(x) = (f\chi_E)(x)$. Thus, $\int_X \phi \le \int_X f\chi_E = \int_E f$.

We thus see that $\int_X \phi \le \int_E f \le \int_X f$. Since $\int_X f - \int_X \phi < \epsilon$, we see that $\int_X f - \int_E f < \epsilon$ as well.

Solution: Let $E_n = \{x \mid f(x) > \frac{1}{n}\}$. We showed in the previous problem that $\mu(E_n) < \infty$ for all n.

Let $f_n = f\chi_{E_n}$. Because $E_1 \subseteq E_2 \subseteq \cdots$, we see that $f_1 \le f_2 \le \cdots$. Moreover, I claim that f_n converges pointwise to f. Indeed, if $f(x) = 0$, then $x \notin E_n$ for all n, so $f_n(x) = 0$ for all n. Thus, we have that $f_n(x) = 0 \to 0 = f(x)$. Meanwhile, if $f(x) > 0$, then there exists an $N > \frac{1}{f(x)}$, where $\frac{1}{\infty} := 0$. Then for all $n \ge N$, we have $f(x) > \frac{1}{N} \ge \frac{1}{n}$, so $x \in E_n$ for all $n \ge N$. Thus $f_n(x) = f(x)$ for all $n \ge N$, so $f_n(x) \to f(x)$.

Thus, we can apply the monotone convergence theorem to conclude that $\int f_n \to \int f$. In particular, for all $\epsilon > 0$, there exists an n such that $\int f - \int f_n < \epsilon$. But recall that $\int f_n = \int f\chi_{E_n} = \int_{E_n} f$ and $\mu(E_n) < \infty$.

5. (a) (5 points) State the monotone convergence theorem.

Solution: Let $f_n \colon X \to [0, \infty]$ be a sequence of nonnegative measurable functions, and let $f_1 \leq f_2 \leq \cdots$. Let $f = \lim_n f_n$. Then f is measurable and

$$\int f = \lim_{n\to\infty} \int f_n.$$

(b) (5 points) State Fatou's Lemma.

Solution: Let $f_n \colon X \to [0, \infty]$ be a sequence of nonnegative measurable functions. Let $f = \liminf_n f_n$. Then f is measurable and

$$\int f \leq \liminf_{n\to\infty} \int f_n.$$

(c) (5 points) State the dominated convergence theorem.

Solution: Let $f_n \colon X \to \mathbb{C}$ be a sequence of integrable functions that converges pointwise almost everywhere to f. Assume that there is a real-valued integrable function g such that $|f_n| \leq g$ for all n. Then f is integrable and

$$\int f = \lim_{n\to\infty} \int f_n.$$

6. (10 points) Give an example of a sequence of functions $f_n\colon \mathbb{R} \to \mathbb{R}$ satisfying the following properties.

   (a) $f_n \in L^1$ for every n.

   (b) $\sum_{n=1}^{\infty} \left(\int f_n\right)$ converges.

   (c) For every x, $\sum_{n=1}^{\infty} f_n(x)$ converges.

   (d) $\sum_{n=1}^{\infty} f_n \in L^1$.

   (e)
$$\int \left(\sum_{n=1}^{\infty} f_n\right) \neq \sum_{n=1}^{\infty} \left(\int f_n\right).$$

   Make sure to justify at least briefly that your example satisfies each of the conditions.

Solution: Let $f_n = \chi_{(n,n+1)} - \chi_{(n+1,n+2)}$. Then f_n is a simple function with positive part $\chi_{(n,n+1)}$ and negative part $\chi_{(n+1,n+2)}$. We see that $\int \chi_{(n,n+1)} = m((n, n+1)) = 1$ and $\int \chi_{(n+1,n+2)} = m((n+1, n+2)) = 1$. Both of these are finite, so $f_n \in L^1$, and, by definition

$$\int f_n = \int f_n^+ - \int f_n^- = 1 - 1 = 0.$$

Thus,

$$\sum_{n=1}^{\infty} \left(\int f_n\right) = \sum_{n=1}^{\infty} 0 = 0,$$

which converges.

Meanwhile, we see that the sum $\sum_{j=1}^{n} f_j$ telescopes, so

$$\sum_{j=1}^{n} f_j = \chi_{(1,2)} - \chi_{(n+1,n+2)}.$$

For every x, we see that $x \notin (n+1, n+2)$ for n large enough, so the sequence $\chi_{(n+1,n+2)}(x)$ converges to zero. Thus, the series $\sum_{j=1}^{\infty} f_j(x)$ converges to $\chi_{(1,2)}(x)$ for all x.

Thus, $\sum_{n=1}^{\infty} f_n = \chi_{(1,2)}$, which is in L^1 because it is nonnegative and $\int \chi_{(1,2)} = m((1,2)) = 1 < \infty$. We now see that

$$\int \left(\sum_{n=1}^{\infty} f_n\right) = 1 \neq 0 = \sum_{n=1}^{\infty} \left(\int f_n\right).$$

Question	Points	Score
1	20	
2	20	
3	20	
4	15	
5	15	
6	10	
Total:	100	