Math 5051 Midterm

October 19, 2018

Name: ______________________________

- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
  - Make a note on the printed page that your work continues on the back of the previous page.
  - On the back of the previous page, put a box around the work that you want graded.
- Give and use definitions from the book or from class.
- You may use any results you remember from the book or from class as long as they are more basic than the result you're asked to prove.

1. (a) (5 points) Let $\mathcal{M}$ be a σ-algebra on X. State the definition of a *measure* μ on $(X, \mathcal{M})$.

   (b) (5 points) State and prove the property of a measure called *monotonicity*.

   (c) (10 points) State and prove the property of a measure called *continuity from below*.

2. Let $(X, \mathcal{M})$ be a measurable space.

   (a) (5 points) Let $f\colon X \to \overline{\mathbb{R}}$. Define what it means for f to be *measurable*.

   (b) (15 points) Let $f_n\colon X \to \overline{\mathbb{R}}$ be a sequence of measurable functions. Show that the function $g\colon X \to \overline{\mathbb{R}}$ defined by $g(x) = \sup_n f_n(x)$ is also measurable.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space, let $f \colon X \to [0, \infty]$ be a measurable function, and assume that $\int f < \infty$.

   (a) (10 points) Show that the set $\{x \mid f(x) = \infty\}$ has measure zero.

   (b) (10 points) Show that the set $\{x \mid f(x) > 0\}$ is σ-finite.

4. (15 points) Let $(X, \mathcal{M}, \mu)$ be a measure space, let $f\colon X \to [0, \infty]$ be a measurable function, and assume that $\int_X f < \infty$.

   Show that you can approximate $\int_X f$ by $\int_E f$ where E has finite measure. That is, for every $\epsilon > 0$, show that there exists an $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_X f - \int_E f < \epsilon$.

5. (a) (5 points) State the monotone convergence theorem.

(b) (5 points) State Fatou's Lemma.

(c) (5 points) State the dominated convergence theorem.

6. (10 points) Give an example of a sequence of functions $f_n\colon \mathbb{R} \to \mathbb{R}$ satisfying the following properties.

   (a) $f_n \in L^1$ for every n.

   (b) $\sum_{n=1}^{\infty} \left(\int f_n\right)$ converges.

   (c) For every x, $\sum_{n=1}^{\infty} f_n(x)$ converges.

   (d) $\sum_{n=1}^{\infty} f_n \in L^1$.

   (e)
$$\int \left(\sum_{n=1}^{\infty} f_n\right) \neq \sum_{n=1}^{\infty} \left(\int f_n\right).$$

   Make sure to justify at least briefly that your example satisfies each of the conditions.

Question	Points	Score
1	20	
2	20	
3	20	
4	15	
5	15	
6	10	
Total:	100	