Math 5051 Final

December 14, 2018

Name: ______________________________

- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
  - Make a note on the printed page that your work continues on the back of the previous page.
  - On the back of the previous page, put a box around the work that you want graded.
- Give and use definitions from the book or from class.
- You may use any results you remember from the book or from class as long as they are more basic than the result you're asked to prove.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let f_n and f be complex-valued measurable functions.

(a) (2 points) Define what it means for f_n to converge to f in L^1.

Solution: The sequence f_n converges to f in L^1 if $\|f - f_n\|_{L^1} \to 0$ as $n \to \infty$, where

$$\|g\|_{L^1} = \int_X |g|\,.$$

(b) (3 points) Define what it means for f_n to converge to f in measure.

Solution: The sequence f_n converges to f in measure if, for all $\epsilon > 0$, we have $\mu(E_{n,\epsilon}) \to 0$ as $n \to \infty$, where

$$E_{n,\epsilon} = \{x \mid |f(x) - f_n(x)| \geq \epsilon\}.$$

(c) (5 points) Show that if f_n converges to f in L^1, then f_n converges to f in measure.

Solution: Observe that $|f - f_n| \geq \epsilon \chi_{E_{n,\epsilon}}$. Integrating both sides, we have that $\|f - f_n\|_{L^1} \geq \epsilon\mu(E_{n,\epsilon})$. Thus, because $\|f - f_n\|_{L^1} \to 0$ as $n \to \infty$ and $\epsilon > 0$, we know that $\mu(E_{n,\epsilon}) \to 0$ as $n \to \infty$ as well.

(d) (5 points) Give and justify an example of functions f_n and f such that f_n converges to f in measure but f_n does not converge to f in L^1.

Solution: Let $f_n = n\chi_{[-1/n,1/n]}$, and let $f = 0$.
We first show that f_n converges to 0 in measure. Here, $E_{n,\epsilon} = \{x \mid |f(x)| \geq \epsilon\}$. For all ϵ, we have that $E_{n,\epsilon} \subseteq [-1/n, 1/n]$, so $\mu(E_{n,\epsilon}) \leq \frac{2}{n}$, and so $\mu(E_{n,\epsilon}) \to 0$ as $n \to \infty$, as desired.
We now show that f_n does not converge to 0 in L^1. Indeed,

$$\|f_n\|_{L^1} = \int f_n = n \cdot \tfrac{2}{n} = 2 \not\to 0$$

as $n \to \infty$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let f_n and f be complex-valued measurable functions.

   (a) (3 points) Define what it means for f_n to converge to f almost uniformly.

   Solution: The sequence f_n converges to f almost uniformly if, for all $\epsilon > 0$, there exists a set E of measure less than ϵ such that f_n converges to f uniformly on E^c.

   (b) (2 points) State Egoroff's Theorem.

   Solution: Assume $\mu(X) < \infty$, and let f_n and f be measurable functions. If f_n converges to f pointwise almost everywhere, then f_n converges to f almost uniformly.

   (c) (5 points) Egoroff's Theorem fails when $X = \mathbb{R}$, $\mathcal{M} = \mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra, and $\mu = m$ is Lebesgue measure. Provide a counterexample to Egoroff's Theorem in this context. That is, give and justify an example of functions f_n and f on $\mathbb{R}$ for which Egoroff's Theorem fails to hold.

   Solution: Let $f_n = \chi_{[n,n+1]}$. For every x, we have $f_n(x) = 0$ for all $n > x$, so $f_n(x) \to 0$ for all x. In other words, f_n converges to 0 pointwise.

   However, f_n does not converge to zero almost uniformly. Let $\epsilon = 1$, and let E be a set of measure less than ϵ. Because $\mu([n, n+1]) = 1$, the interval $[n, n+1]$ cannot be a subset of E by monotonicity. Thus, for all n, there exists an $x \in [n, n+1] \cap E^c$. In other words, for all n, there exists an $x \in E^c$ such that $f_n(x) = 1$. That means that $\sup_{x \in E^c} |f_n(x)| = 1$, but uniform convergence requires that $\sup_{x \in E^c} |f_n(x)|$ converges to zero. Thus, f_n does not converge to 0 uniformly on E^c for any choice of E with $\mu(E) < 1$.

3. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces.

(a) (3 points) Define the product σ-algebra $\mathcal{M} \otimes \mathcal{N}$ on $X \times Y$.

Solution: The product σ-algebra $\mathcal{M} \otimes \mathcal{N}$ is the σ-algebra generated by

$$\{E \times F \mid E \in \mathcal{M}, F \in \mathcal{N}\}.$$

(b) (12 points) State the Fubini and Tonelli Theorems.

Solution: Assume that μ and ν are σ-finite.
The Tonelli theorem considers a function $f \in L^+(X \times Y)$. It tells us that the functions $f_x \colon Y \to \mathbb{R}$ and $f^y \colon X \to \mathbb{R}$ defined by $f_x(y) = f(x, y) = f^y(x)$ are in $L^+(Y)$ and $L^+(X)$, respectively, and the functions $x \mapsto \int_Y f_x \, d\nu$ and $y \mapsto \int_X f^y \, d\mu$ are in $L^+(X)$ and $L^+(Y)$, respectively. Moreover

$$\int_X \left(\int_Y f_x(y) \, d\nu(y) \right) d\mu(x) = \int_{X \times Y} f \, d(\mu \times \nu) = \int_Y \left(\int_X f^y(x) \, d\mu(x) \right) d\nu(y).$$

The Fubini theorem considers a function $f \in L^1(X \times Y)$. It tells us that the function $f_x \colon Y \to \mathbb{C}$ is in $L^1(Y)$ for almost every x and $f^y \colon X \to \mathbb{C}$ is in $L^1(X)$ for almost every y. Moreover, the functions $x \mapsto \int_Y f_x \, d\nu$ and $y \mapsto \int_X f^y \, d\mu$ are in $L^1(X)$ and $L^1(Y)$, respectively, and we once again have

$$\int_X \left(\int_Y f_x(y) \, d\nu(y) \right) d\mu(x) = \int_{X \times Y} f \, d(\mu \times \nu) = \int_Y \left(\int_X f^y(x) \, d\mu(x) \right) d\nu(y).$$

4. Let $(X, \mathcal{M})$ be a measurable space.

   (a) (3 points) State the Hahn Decomposition Theorem

   Solution: Let ν be a signed measure. There exists a decomposition $X = P \sqcup N$, where P is positive for ν, and N is negative for ν. Moreover, if $X = P' \sqcup N'$ is another such decomposition, then $P \Delta P' = N \Delta N'$ is null for ν.

   (b) (2 points) State the Jordan Decomposition Theorem

   Solution: Let ν be a signed measure. Then $\nu = \nu^+ - \nu^-$, where ν^+ and ν^- are positive measures that are mutually singular. This decomposition is unique.

5. Let $(X, \mathcal{M})$ be a measurable space. Let μ be a positive measure, and let ν be a signed measure.

   (a) (3 points) Define mutual singularity.

   Solution: The measures μ and ν are mutually singular, denoted $\mu \perp \nu$, if there exists a decomposition $X = E \sqcup F$ such that E is null for ν and F is null for μ.

   (b) (2 points) Define absolute continuity.

   Solution: The measure ν is absolutely continuous with respect to μ, denoted $\nu \ll \mu$, if $\mu(E) = 0$ implies $\nu(E) = 0$ for all $E \in \mathcal{M}$.

   (c) (5 points) State the Lebesgue-Radon-Nikodym Theorem.

   Solution: If μ and ν are σ-finite, then there exists a Lebesgue decomposition $\nu = \lambda + \rho$, where λ and ρ are σ-finite signed measures, $\lambda \perp \nu$, and $\rho \ll \mu$. Moreover, there exists a extended μ-integrable density function f such that $\rho(E) = \int_E f \, d\mu$ for all $E \in \mathcal{M}$. The signed measures λ and ρ are unique, and f is μ-a.e.-unique. That is, if we also have $\rho(E) = \int_E f' \, d\mu$, then $f = f'$ μ-a.e.

6. Let $(X, \mathcal{M})$ be a measurable space.

(a) (10 points) Let μ be a positive measure, and let E_j be a sequence of measurable sets. Assume that $\sum_{j=1}^{\infty} \mu(E_j) < \infty$. Show that $\mu(\limsup E_j) = 0$.
Recall that $\limsup E_j = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j$.

Solution: Let $F_n = \bigcup_{j=n}^{\infty} E_j$, so $\limsup E_j = \bigcap_{n=1}^{\infty} F_n$. Then, by countable subadditivity,

$$\mu(F_n) \leq \sum_{j=n}^{\infty} \mu(E_j).$$

If $\sum_{j=1}^{\infty} \mu(E_j) < \infty$, then the tails $\sum_{j=n}^{\infty} \mu(E_j)$ converge to zero as $n \to \infty$. Indeed, $\sum_{j=n}^{\infty} \mu(E_j) = \sum_{j=1}^{\infty} \mu(E_j) - \sum_{j=1}^{n-1} \mu(E_j)$, which converges to zero by the definition of infinite series. We conclude that $\mu(F_n) \to 0$ as $n \to \infty$.

We now apply continuity from above. We indeed have $F_1 \supseteq F_2 \supseteq \cdots$, and we note that the finiteness hypothesis $\mu(F_1) \leq \sum_{j=1}^{\infty} \mu(E_j) < \infty$ is satisfied. Hence $\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = 0$, as desired.

(b) (15 points) Let μ and ν be positive measures. Show that $\nu \perp \mu$ if and only if there exists a sequence E_j of measurable sets such that $\mu(E_j) \to 0$ and $\nu(E_j^c) \to 0$ as $j \to \infty$.
Hint: Construct a subsequence of the E_j so that you can apply part (a).

Solution: For the easy direction, assume that $\nu \perp \mu$. Then $X = E \sqcup F$, where $\mu(E) = 0$ and $\nu(F) = 0$. We simply set $E_j = E$. Then $\mu(E_j) = 0 \to 0$, and $\nu(E_j^c) = \nu(F) = 0 \to 0$.

Using the definition of $\mu(E_j) \to 0$ and $\nu(E_j^c) \to 0$, for each natural number k, we can choose a j_k sufficiently large so that $\mu(E_{j_k})$ and $\nu(E_{j_k}^c)$ are both smaller than 2^{-k}. Therefore, $\sum_{k=1}^{\infty} \mu(E_{j_k}) = 1 < \infty$ and $\sum_{k=1}^{\infty} \nu(E_{j_k}^c) = 1 < \infty$, so by part (a) we conclude that $\mu(\limsup E_{j_k}) = 0$ and $\nu(\limsup E_{j_k}^c) = 0$.

Thus, we let $E = \limsup E_{j_k}$, so $\mu(E) = 0$. To show $\nu(E^c) = 0$, we can use

$$E^c = \liminf E_{j_k}^c \subseteq \limsup E_{j_k}^c.$$

Since $\limsup E_{j_k}^c$ is null for ν, so is E^c. Thus, $\nu \perp \mu$, as desired.

Alternatively, we can show that $\nu(E^c) = 0$ by computing that

$$E^c = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_{j_k}^c.$$

To show that $\nu(E^c) = 0$, it suffices to show that $\nu\left(\bigcap_{k=n}^{\infty} E_{j_k}^c\right) = 0$ for all n. By monotonicity,

$$\nu\left(\bigcap_{k=n}^{\infty} E_{j_k}^c\right) \leq \nu\left(E_{j_k}^c\right)$$

for all $k \geq n$. We know that $\nu(E_{j_k}^c) \to 0$ as $k \to \infty$, so taking the limit as $k \to \infty$, we find that $\nu\left(\bigcap_{k=n}^{\infty} E_{j_k}^c\right) = 0$, as desired. Note that in this argument, we did not actually need the subsequence to satisfy $\sum_{k=1}^{\infty} \nu(E_{j_k}^c) < \infty$.

7. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $\mathcal{N}$ be a sub-σ-algebra of $\mathcal{M}$, and let ν be the restriction of μ to $\mathcal{N}$.

   (a) (10 points) Given a $\mathcal{M}$-measurable function $f \in L^1(\mu)$, show that there exists an $\mathcal{N}$-measurable function $g \in L^1(\nu)$ such that $\int_E f\,d\mu = \int_E g\,d\nu$ for all $E \in \mathcal{N}$. This function is called the *conditional expectation* of f on $\mathcal{N}$.

   Solution: Define a signed measure ρ on $\mathcal{N}$ by $\rho(E) = \int_E f\,d\mu$. We can check that ρ is indeed a signed measure. We see that $\rho(E)$ is finite because $f \in L^1(\mu)$. Moreover, we can check countable additivity because if $E = E_1 \sqcup E_2 \sqcup \cdots$, we let $f_j = f\chi_{E_j}$, so, by definition, $\rho(E_j) = \int f_j\,d\mu$. Then $\sum f_j = f\chi_E$ and $\sum |f_j| = |f\chi_E|$. For positive functions, we know that $\sum \int |f_j|\,d\mu = \int |f\chi_E|\,d\mu < \infty$. One of the consequences of the dominated convergence theorem that we proved in class then tells us that $\sum \int f_j\,d\mu = \int f\chi_E\,d\mu$. In other words, $\sum \rho(E_j) = \rho(E)$.
   Moreover, $\rho \ll \nu$. Indeed, if $E \in \mathcal{N}$ and $\nu(E) = 0$, then $\mu(E) = 0$, so $\int_E f\,d\mu = 0$. We also have that ρ and ν are finite, and hence σ-finite. Thus, there exists a Radon-Nikodym derivative, that is, a $\mathcal{N}$-measurable function g such that $\rho(E) = \int_E g\,d\nu$ for all $E \in \mathcal{N}$. In other words, $\int_E f\,d\mu = \int_E g\,d\nu$.
   To check that $g \in L^1(\nu)$, we must check that the integrals $\int g^+\,d\nu$ and $\int g^-\,d\nu$ are finite. Let $E = \{x \mid g(x) \geq 0\}$. We know that $E \in \mathcal{N}$ by definition of $\mathcal{N}$-measurability. Using the fact that $f \in L^1(\mu)$, we then have that $\int g^+\,d\nu = \int_E g\,d\nu = \int_E f\,d\mu < \infty$. Similarly, $\int g^-\,d\nu = -\int_{E^c} g\,d\nu = -\int_{E^c} f\,d\mu < \infty$.

   Let X be the finite set $\{1, 2, 3, 4, 5, 6\}$, let $\mathcal{M} = \mathcal{P}(X)$ be the power set of X, and let μ be counting measure. Let $f\colon X \to \mathbb{R}$ be defined by $f(n) = n$.

   (b) (5 points) Explicitly write down the function g constructed above if $\mathcal{N}$ is the σ-algebra $\{\emptyset, X\}$. Make sure that g is $\mathcal{N}$-measurable.

   Solution: The condition we seek to satisfy is $\int_X g\,d\nu = \int_X f\,d\mu$. (The condition when $E = \emptyset$ is trivial.) Because we're working with a finite set and counting measure, this really means $\sum_{n=1}^6 g(n) = \sum_{n=1}^6 f(n) = 21$.
   In this case, g being $\mathcal{N}$-measurable implies that for all $z \in \mathbb{R}$, either $g^{-1}(\{z\}) = X$ or $g^{-1}(\{z\}) = \emptyset$. This implies that g is a constant function. Thus, we seek a constant function such that $\sum_{n=1}^6 g(n) = 21$. We conclude that $g(n) = 3.5$ for $n = 1, \ldots, 6$.
   Note that if f represents the value of a six-sided die roll, then g tells us the expected value of the number we will see on the die.

   (c) (5 points) Explicitly write down the function g constructed above if $\mathcal{N}$ is the σ-algebra $\{\emptyset, \{1, 3, 5\}, \{2, 4, 6\}, X\}$. Make sure that g is $\mathcal{N}$-measurable.

   Solution: Following the above reasoning, we see that, in this case, g being $\mathcal{N}$-measurable means that $g(1) = g(3) = g(5)$ and $g(2) = g(4) = g(6)$. Indeed, we can check, for example, that measurability implies $g^{-1}(\{g(1)\}) \in \mathcal{N}$, and we have $1 \in g^{-1}(\{g(1)\}$, which implies that $\{1, 3, 5\} \subseteq g^{-1}(\{g(1)\})$.
   The requirement that $\int_E g\,d\nu = \int_E f\,d\mu$ then tells us that $g(1) + g(3) + g(5) = f(1) + f(3) + f(5) = 9$ and $g(2) + g(4) + g(6) = f(2) + f(4) + f(6) = 12$. The requirement

that g be constant on these sets then implies that $g(1) = g(3) = g(5) = 3$ and $g(2) = g(4) = g(6) = 4$.

The function g is $\mathcal{N}$-measurable because it a simple function with respect to $\mathcal{N}$, namely $3\chi_{\{1,3,5\}} + 4\chi_{\{2,4,6\}}$.

If we want to, we can check explicitly that g is $\mathcal{N}$-measurable. Let Z be a measurable subset of $\mathbb{R}$. If Z contains 3 but not 4, then $g^{-1}(Z) = \{1, 3, 5\} \in \mathcal{N}$, and similarly if Z contains 4 but not 3. If Z contains neither, then $g^{-1}(Z) = \emptyset$, and if Z contains both, then $g^{-1}(Z) = X$.

Note that, here, the value of $g(1) = g(3) = g(5)$ tells us the expected value of the die roll if we already know that the die roll is odd. Likewise, the value of $g(2) = g(4) = g(6)$ tells us the expected value of the die roll conditioned on the knowledge that it is even.

Question	Points	Score
1	15	
2	10	
3	15	
4	5	
5	10	
6	25	
7	20	
Total:	100	