Math 5051 Final

December 14, 2018

Name: ______________________________

- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
  - Make a note on the printed page that your work continues on the back of the previous page.
  - On the back of the previous page, put a box around the work that you want graded.
- Give and use definitions from the book or from class.
- You may use any results you remember from the book or from class as long as they are more basic than the result you're asked to prove.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let f_n and f be complex-valued measurable functions.
   (a) (2 points) Define what it means for f_n to converge to f in L^1.

   (b) (3 points) Define what it means for f_n to converge to f in measure.

   (c) (5 points) Show that if f_n converges to f in L^1, then f_n converges to f in measure.

   (d) (5 points) Give and justify an example of functions f_n and f such that f_n converges to f in measure but f_n does not converge to f in L^1.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let f_n and f be complex-valued measurable functions.

   (a) (3 points) Define what it means for f_n to converge to f almost uniformly.

   (b) (2 points) State Egoroff's Theorem.

   (c) (5 points) Egoroff's Theorem fails when $X = \mathbb{R}$, $\mathcal{M} = \mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra, and $\mu = m$ is Lebesgue measure. Provide a counterexample to Egoroff's Theorem in this context. That is, give and justify an example of functions f_n and f on $\mathbb{R}$ for which Egoroff's Theorem fails to hold.

3. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces.

   (a) (3 points) Define the product σ-algebra $\mathcal{M} \otimes \mathcal{N}$ on $X \times Y$.

   (b) (12 points) State the Fubini and Tonelli Theorems.

4. Let $(X, \mathcal{M})$ be a measurable space.

   (a) (3 points) State the Hahn Decomposition Theorem

   (b) (2 points) State the Jordan Decomposition Theorem

5. Let $(X, \mathcal{M})$ be a measurable space. Let μ be a positive measure, and let ν be a signed measure.
   (a) (3 points) Define mutual singularity.

   (b) (2 points) Define absolute continuity.

   (c) (5 points) State the Lebesgue-Radon-Nikodym Theorem.

6. Let $(X, \mathcal{M})$ be a measurable space.

   (a) (10 points) Let μ be a positive measure, and let E_j be a sequence of measurable sets. Assume that $\sum_{j=1}^{\infty} \mu(E_j) < \infty$. Show that $\mu(\limsup E_j) = 0$.
   Recall that $\limsup E_j = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j$.

   (b) (15 points) Let μ and ν be positive measures. Show that $\nu \perp \mu$ if and only if there exists a sequence E_j of measurable sets such that $\mu(E_j) \to 0$ and $\nu(E_j^c) \to 0$ as $j \to \infty$.
   Hint: Construct a subsequence of the E_j so that you can apply part (a).

7. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $\mathcal{N}$ be a sub-σ-algebra of $\mathcal{M}$, and let ν be the restriction of μ to $\mathcal{N}$.

   (a) (10 points) Given a $\mathcal{M}$-measurable function $f \in L^1(\mu)$, show that there exists an $\mathcal{N}$-measurable function $g \in L^1(\nu)$ such that $\int_E f\, d\mu = \int_E g\, d\nu$ for all $E \in \mathcal{N}$. This function is called the *conditional expectation* of f on $\mathcal{N}$.

   Let X be the finite set $\{1, 2, 3, 4, 5, 6\}$, let $\mathcal{M} = \mathcal{P}(X)$ be the power set of X, and let μ be counting measure. Let $f\colon X \to \mathbb{R}$ be defined by $f(n) = n$.

   (b) (5 points) Explicitly write down the function g constructed above if $\mathcal{N}$ is the σ-algebra $\{\emptyset, X\}$. Make sure that g is $\mathcal{N}$-measurable.

   (c) (5 points) Explicitly write down the function g constructed above if $\mathcal{N}$ is the σ-algebra $\{\emptyset, \{1, 3, 5\}, \{2, 4, 6\}, X\}$. Make sure that g is $\mathcal{N}$-measurable.

Question	Points	Score
1	15	
2	10	
3	15	
4	5	
5	10	
6	25	
7	20	
Total:	100	