Topology I Midterm 1

October 5, 2016

Name: ______________________________

- There are five questions, each worth twenty points, so you should pace yourself at around 10–12 minutes per question, since they vary in difficulty and you'll want to check your work.
- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
  - Make a note on the printed page that your work continues on the back of the previous page.
  - On the back of the previous page, put a box around the work that you want graded.

1. (a) (6 points) Define the standard topology on $\mathbb{R}^n$. A set U is open in $\mathbb{R}^n$ if ...

   (b) (7 points) Show from the above definition that the interval (a, b) is open in $\mathbb{R}$.

   (c) (7 points) Show that a subset of $\mathbb{R}$ is open if and only if it is the union of open intervals.

2. (a) (6 points) Let $f\colon X \to Y$ be a function between topological spaces X and Y. Define continuity. The function f is continuous if ...

   (b) (6 points) Let A be a subset of a topological space X. Define the subspace topology on A.

   (c) (8 points) Let $f\colon X \to Y$ be a continuous map. Give the image $f(X) \subseteq Y$ the subspace topology. Restrict the codomain by letting $\hat{f}\colon X \to f(X)$ be defined by $\hat{f}(x) = f(x)$. Show that $\hat{f}$ is continuous.

3. (a) (6 points) Let $\sim$ be an equivalence relation on a topological space X. Define the quotient topology on $X/{\sim}$.

(b) (6 points) Let X be a set. Define the discrete topology on X.

(c) (8 points) Let X have the discrete topology. Show that the quotient topology on $X/{\sim}$ is also discrete.

4. (a) (6 points) Let X be a topological space, and let x_i be a sequence in X. Define convergence. The sequence x_i converges to a point $x \in X$ if ...

(b) (6 points) Let A be a subset of a topological space X. Define the closure $\overline{A}$ of A.

(c) (8 points) Let $x_i \in A$ be a sequence that converges to a point $x \in X$. Show that $x \in \overline{A}$. Conclude that, if A is closed, then A is sequentially closed, where sequentially closed means that if a sequence $x_i \in A$ has a limit x, then $x \in A$.

5. (a) (7 points) Let $x \in \mathbb{R}^n$. For $i \in \mathbb{N}$, let $U_i = B_{1/i}(x)$, the open ball of radius $1/i$ around x. For each i, let $x_i \in U_i$. Show directly from the definition of the standard topology on $\mathbb{R}^n$ and the definition of sequence convergence in a topological space that x_i converges to x.

(b) (7 points) For $A \subseteq \mathbb{R}^n$, prove the converse of question 4. That is, let $x \in \overline{A}$. Show that there is a sequence $x_i \in A$ that converges to x.

(c) (6 points) Conclude that if $A \subseteq \mathbb{R}^n$ is sequentially closed, then A is closed, where sequentially closed was defined in question 4.

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	