TOPOLOGY HOMEWORK 9, DUE NOVEMBER 2

Munkres's exercises on the product topology are good, so that's what's going on this week.

(1) Do exercise 19.1 on page 118 of Munkres. If you'd like, you may prove it only for the product topology and skip the box topology, though the box topology is a good warmup.

(2) Do exercise 19.2 on page 118 of Munkres. Again, feel free to only do the product topology. The question is asking whether the product topology on $\prod A_\alpha$ matches the subspace topology as a subspace of $\prod X_\alpha$.

(3) Do exercise 19.3 on page 118 of Munkres.

(4) Do exercise 19.4 on page 118 of Munkres. Suggestion: Show that $(X_1 \times \cdots \times X_{n-1}) \times X_n$ satisfies the universal property for $X_1 \times \cdots \times X_n$, using the universal property for $X_1 \times \cdots \times X_{n-1}$ and for $Y \times X_n$. There are several other tactics that will work, but do keep in mind that most open sets in the product topology are not basis elements.

(5) Do exercise 19.5 on page 118 of Munkres.

(6) (a) Recall the definition of convergence of a sequence in the middle of page 98 of Munkres. Prove the basis criterion for the convergence of a sequence: Let X be a topological space with basis $\mathcal{B}$. Show that a sequence x_n converges to x in X if for all basis elements B containing x, there is an $N \in \mathbb{N}$ such that $x_n \in B$ for all $n \geq N$.

(b) Prove the subbasis criterion for the convergence of a sequence: Let X be a topological space with subbasis $\mathcal{S}$. Show that a sequence x_n converges to x in X if for all subbasis elements S containing x, there is an $N \in \mathbb{N}$ such that $x_n \in S$ for all $n \geq N$. Warning: Make sure your proof fails if you use arbitrary intersections.

(c) Do exercise 19.6 on page 118 of Munkres. Make sure to answer the question about the box topology, providing a proof or counterexample for each direction of the implication.

(7) Do exercise 19.7 on page 118 of Munkres.

(8) Do exercise 19.8 on page 118 of Munkres.

(9) Do exercise 19.10 on page 118 of Munkres. Suggestion: For part (a), use exercise 13.4(b). Hint: For part (d), to start thinking about the problem, consider the special case where each X_α is $\mathbb{R}$, the set A is $\mathbb{R}$ also, and each $f_\alpha \colon A \to X_\alpha$ is the identity map $\mathbb{R} \to \mathbb{R}$. In this case, what does Z look like? First, think about this example with a two-element index set, so the product is just $\mathbb{R} \times \mathbb{R}$, which you can graph, and then think about it with a general index set. Then adapt that proof to the general setting of arbitrary spaces X_α and A. Warning: Unlike inverse images, direct images preserve unions but not intersections, so if you set it up right you can work with a basis, but it won't work very well to work with a subbasis for part (d).