TOPOLOGY HOMEWORK 2, DUE SEPTEMBER 14

Instructions

(1) Write down the names of the people you worked with.
(2) Write down any resources you used other than ones that most of your classmates would be familiar with, such as the textbook or Wikipedia.
(3) Write down the number of hours it took you to complete this assignment.
(4) Write your name, Math 4171, and the homework number.
(5) Hand in your homework in class.

Problems

(1) We briefly discussed the image (direct image) and preimage (inverse image) at the end of class on Wednesday. You can review it on page 19 of the book. Then do exercise 2.1 on page 20 of Munkres.
(2) We'll expand a bit on exericse 2.1
  (a) Let $f\colon A \to B$ be a function that is *not* injective. Find a subset $A_0 \subseteq A$ such that $A_0 \neq f^{-1}(f(A_0))$.
  (b) Let $f\colon A \to B$ be a function that is *not* surjective. Find a subset $B_0 \subseteq B$ such that $f(f^{-1}(B_0)) \neq B_0$.
(3) Do exercise 2.2 on page 20 of Munkres.
(4) Again, we'll expand on this exercise. Assume that $f\colon A \to B$ is *not* injective.
  (a) Find subsets A_0 and A_1 such that $f(A_0 \cap A_1) \neq f(A_0) \cap f(A_1)$.
  (b) Find subsets A_0 and A_1 such that $f(A_0 \setminus A_1) \neq f(A_0) \setminus f(A_1)$.
(5) Do exercise 2.3 on page 21 of Munkres. If you are brave, you can write in exercise 2.2 that the claims follow from exercise 2.3, but then your points on exercise 2.2 rely on getting exercise 2.3 correct.
(6) Do exercise 2.4 on page 21 of Munkres.
(7) Do exercise 2.5 on page 21 of Munkres.
(8) Do exercise 16.1 on page 91 of Munkres. A way to think of the questions is that there are two natural ways to define a topology on A. The easiest one is that $A \subseteq X$, so we can give it the subspace topology. The more complex one is that first we can first give $Y \subseteq X$ the subspace topology. Then we can forget about X. Since Y is a topological space, and $A \subseteq Y$, we can give A the subspace topology using the topology on Y. It would be annoying if these two constructions gave us different topologies on A. The exercise asks you to show that it's okay.
(9) Do exercise 17.2 on page 100 of Munkres. The statement, "A is closed in Y" means what you might expect, namely that A is a subset of Y that is closed with respect to the subspace topology on Y.
(10) Do exercise 17.4 on page 100 of Munkres.
(11) Do exercise 17.12 on page 101 of Munkres. Again, subspaces will have the subspace topology unless otherwise stated.
(12) This question is about the *separation axioms*, some of which are listed below.

- In class, we defined a space to be T_0 (Kolmogorov) if, for any two distinct points x and y, there is either an open set that contains x but not y, *or* an open set that contains y but not x.
- A space is T_1 (accessible or Fréchet) if, for any two distinct points x and y, there is an open set that contains x but not y *and* an open set that contains y but not x.
- In class we defined a space to be T_2 (Hausdorff) if, for any two distinct points x and y, there are disjoint open neighborhoods U and V of x and y, respectively.
- A space is T_3 (regular Hausdorff) if it is T_0 and, for any closed set C and any $x \notin C$, there are disjoint open neighborhoods of x and C. That is, there are disjoint open sets U and V such that $x \in U$ and $C \subseteq V$.
- A space is T_4 (normal Hausdorff) if it is T_1 and, for any disjoint closed sets C and D, there are disjoint open neighborhoods of C and D. That is, there are disjoint open sets U and V such that $C \subseteq U$ and $D \subseteq V$.

(a) Find a space that is not T_0. Justify your claim. Hint: We've seen an example.

(b) Find a space that is T_0 but not T_1. Justify your claim. Hint: See page 76 of Munkres.

(c) Find a space that is T_1 but not T_2. Justify. Hint: You've seen an example.

(d) It is clear from the definitions that if a space is T_2, then it is T_1, and if a space is T_1, then it is T_0. Show that if a space is T_3, then it is T_2.

(e) One might wonder why the definition of T_3 assumes that the space is T_0. Find a space that is not T_0 that would otherwise satisfy the definition of T_3. Justify your claim. Hint: Chances are, your example from part 12a will work.

(f) Show that X is T_1 if and only if for any $x \in X$, the singleton set $\{x\}$ is closed. Conclude that T_4 implies T_3.

(g) One might wonder if the T_1 assumption in the definition of T_4 is necessary. Find a T_0 space that is not T_1 that would otherwise satisfy the definition of T_4. Justify your claim. Hint: Chances are, your example from part 12b will work.

(13) In this problem, we will show that $\mathbb{R}$ with the standard topology is T_4.

(a) Let C and D be disjoint closed subsets of $\mathbb{R}$. For any $c \in C$, show that there exists an ε_c such that the ball $B_{\varepsilon_c}(c)$ is disjoint from D. Similarly, for any $d \in D$, there is a ball $B_{\varepsilon_d}(d)$ is disjoint from C.

(b) If we naively took the unions of these open balls, the resulting open sets might not be disjoint. Also, we haven't really used the notion of distance, which will have to come into play, otherwise we might accidentally prove the false claim that every topological space is T_4. The key step is to halve the radii of the balls, and define

$$U = \bigcup_{c \in C} B_{\varepsilon_c/2}(c), \qquad V = \bigcup_{d \in D} B_{\varepsilon_d/2}(d).$$

Show that U and V are disjoint open neighborhoods of C and D.

(c) Check that $\mathbb{R}$ is T_1, and conclude that $\mathbb{R}$ is T_4.