Differential Equations (Math 217) Practice Midterm 3

November 15, 2016

- No calculators, notes, or other resources are allowed.
- There are 14 multiple-choice questions, worth 5 points each, and two hand-graded questions, worth 15 points each, for a total of 100 points.
- For the hand-graded questions, please turn in your solution to the two questions to separate piles.
- Write your name and student ID and circle your section on each page of your solutions to the hand-graded questions. There are two questions, so you will do this two times.

The practice midterm and the actual midterm might cover material from *any* of the following:

- Lectures through November 9.
- Chapter 1 and sections 3.1 through 3.6 of the textbook.
- Homework and Webwork through homework and Webwork 11.

- The hyperbolic trigonometric functions are defined by

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}),$$
$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}).$$

- You can compute other logarithms using $\ln(mn) = \ln m + \ln n$ or by interpolating between two known values.

$$\ln 2 \approx 0.69,$$
$$\ln 3 \approx 1.10,$$
$$\ln 5 \approx 1.61,$$
$$\ln 7 \approx 1.95,$$
$$\ln 11 \approx 2.40.$$

- Other approximations:
  - $\pi \approx 3$.
  - $g \approx 10\,\mathrm{m/s^2}$.

- If n is positive, to determine whether $n < \sqrt{x}$, determine whether $n^2 < x$.

- The method of variation of parameters involves the system of equations

$$u_1' y_1 + u_2' y_2 = 0,$$
$$u_1' y_1' + u_2' y_2' = f(x).$$

1. Determine the amplitude of the solution to the initial value problem

$$y'' + 4y = 0, \qquad y(0) = -4, \qquad y'(0) = -8.$$

Decide whether the amplitude is:

A. Between zero and one.
B. Between one and two.
C. Between two and three.
D. Between three and four.
E. Between four and five.
F. Between five and six.
G. Between six and seven.
H. Between seven and eight.
I. Between eight and nine.
J. Between nine and ten.
K. Between ten and eleven.

2. Find a solution y_p to the differential equation

$$y^{(3)} - y'' - 3y' = 12xe^x$$

that is either the product of a polynomial and an exponential, or the product of a polynomial, an exponential, and a sinusoid, keeping in mind that a constant polynomial counts as a polynomial. Evaluate $y_p'(0)$.

A. $-\frac{3}{2}$.
B. $-\frac{4}{3}$.
C. -24.
D. -12.
E. $-\frac{24}{19}$.
F. $\frac{12}{5}$.
G. $-\frac{12}{7}$.
H. $-\frac{8}{5}$.
I. $-\frac{6}{5}$.
J. $\frac{6}{5}$.
K. $\frac{8}{5}$.

3. Find a solution y_p to the differential equation

$$y^{(3)} - y'' - 3y' = 12x$$

that is either the product of a polynomial and an exponential, or the product of a polynomial, an exponential, and a sinusoid, keeping in mind that a constant polynomial counts as a polynomial. Evaluate $y_p'(0)$.

A. $\frac{3}{2}$.

B. $\frac{4}{3}$.

C. 24.

D. 12.

E. $\frac{24}{19}$.

F. $-\frac{12}{5}$.

G. $\frac{12}{7}$.

H. $\frac{8}{5}$.

I. $\frac{6}{5}$.

J. $-\frac{6}{5}$.

K. $-\frac{8}{5}$.

4. A *smart fluid* is a fluid whose viscosity can be changed by applying a magnetic field. Consequently, if you have a dashpot filled with a smart fluid, you can change its damping constant c in real time. Such an adjustable dashpot has several applications, including in the shock absorbers of high-end cars. Assume that the car has a mass of 2000 kg. Meanwhile, the suspension acts as a spring with spring constant $k = 50\,000\,\mathrm{N/m}$. You'd like to set the viscosity of the smart fluid so that, after hitting a bump, the car crosses its equilibrium position at most once. Of these choices, with which value(s) of c will that happen?

    I. $c = 10\,000\,\mathrm{N/(m/s)}$.
    II. $c = 20\,000\,\mathrm{N/(m/s)}$.
    III. $c = 30\,000\,\mathrm{N/(m/s)}$.
    IV. $c = 40\,000\,\mathrm{N/(m/s)}$.
    V. $c = 50\,000\,\mathrm{N/(m/s)}$.

    A. I.
    B. I and II.
    C. I, II, and III.
    D. I, II, III, and IV.
    E. I, II, III, IV, and V.
    F. II, III, IV, and V.
    G. III IV, and V.
    H. IV and V.
    I. V.
    J. None of them.

5. Consider the periodic motion

$$x(t) = 2\sqrt{3}\cos(5t) + 2\sin(5t).$$

At what time does the third maximum of x occur, starting at $t = 0$?

A. $t = \frac{1}{6}\pi$.
B. $t = \frac{5}{6}\pi$.
C. $t = \frac{7}{6}\pi$.
D. $t = \frac{11}{6}\pi$.
E. $t = \frac{13}{6}\pi$.
F. $t = \frac{17}{6}\pi$.
G. $t = \frac{19}{6}\pi$.
H. $t = \frac{23}{6}\pi$.
I. $t = \frac{25}{6}\pi$.
J. $t = \frac{29}{6}\pi$.
K. $t = \frac{31}{6}\pi$.

6. Consider a mass-spring-dashpot system with $m = 1\,\text{kg}$, $c = 3\,\text{N}/(\text{m}/\text{s})$, and $k = 50\,\text{N}/\text{m}$ under the influence of an external force $F(t) = (7\,\text{N})\cos(5t)$, where t is in seconds. What is the *approximate* amplitude of the steady periodic response of this mass-spring-dashpot system to the external force?

   A. 1.0 cm.
   B. 2.5 cm.
   C. 5.0 cm.
   D. 10 cm.
   E. 25 cm.
   F. 50 cm.
   G. 1.0 m.
   H. 2.5 m.
   I. 5.0 m.
   J. 10 m.
   K. 25 m.

7. On the moon, the acceleration due to gravity is about $1.6\,\mathrm{m/s^2}$. If a pendulum takes one second to complete one swing back and forth (a full period) on Earth, how long would it take to complete one swing back and forth on the moon?

   A. 0.010 seconds.
   B. 0.025 seconds.
   C. 0.050 seconds.
   D. 0.10 seconds.
   E. 0.25 second.
   F. 0.50 seconds.
   G. 1.0 seconds.
   H. 2.5 seconds.
   I. 5.0 seconds.
   J. 10 seconds.
   K. 25 seconds.

8. We expect that a golf ball attached to a string will swing back and forth like a pendulum, but a feather of the same size attached to a string will get stopped by air resistance and not swing back and forth. Assume that for an object of this size travelling at $1\,\mathrm{m/s}$, the force of air resistance is $4 \times 10^{-3}\,\mathrm{N}$, and that this force scales proportionally to speed.

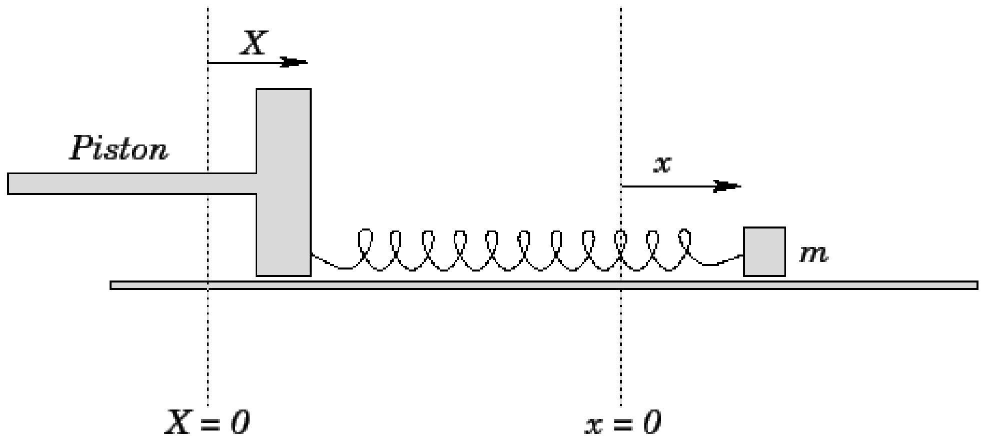
   Assume that the length of the string is $2.5\,\mathrm{m}$. What is the minimum mass required of the object attached to the string so that the object swings back and forth? More precisely, find the value for the mass so than any heavier object will oscillate back and forth, and any lighter object will not.

   Note: Be mindful of the fact that speed is the size of the rate of change of displacement and thus has units m/s.

   A. 1 g.
   B. 2 g.
   C. 5 g.
   D. 10 g.
   E. 20 g.
   F. 50 g.
   G. 100 g.
   H. 200 g.
   I. 500 g.
   J. 1 kg.
   K. 2 kg.

   Note: In reality, the force of air resistance is proportional to the square of the velocity, so this model is actually more apropriate for a pendulum immersed in a viscous fluid.

9. Consider the following forced harmonic oscillator with positive but negligible damping.



When the piston does not move, if we set the mass m in motion, it oscillates at a frequency of 6 Hz. When the piston oscillates sinusoidally with an amplitude of 20 cm, we observe that the steady periodic oscillations of the mass have a slightly smaller amplitude of 16 cm. What is the frequency of the oscillations of the piston?

A. 1 Hz.

B. 2 Hz.

C. 3 Hz.

D. 4 Hz.

E. 5 Hz.

F. 6 Hz.

G. 7 Hz.

H. 8 Hz.

I. 9 Hz.

J. 10 Hz.

K. 11 Hz.

10. The answer choices below represent families of functions. For one of the choices, there is a solution to the differential equation

$$y''' + 6y'' + 11y' + 6y = 12e^x$$

that is *not* a member of the family. Which choice is it?

A. $\frac{1}{2}e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(2e^{-x} + e^{-3x}) + C_3(e^{-2x} + e^{-3x})$.
B. $\frac{1}{2}e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(2e^{-x} + e^{-3x}) + C_3(e^{-2x} + 2e^{-3x})$.
C. $\frac{1}{2}e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(2e^{-x} + e^{-3x}) + C_3(e^{-2x} + 4e^{-3x})$.
D. $\frac{1}{2}e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(2e^{-x} + e^{-3x}) + C_3(2e^{-2x} + e^{-3x})$.
E. $\frac{1}{2}e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(2e^{-x} + e^{-3x}) + C_3(4e^{-2x} + e^{-3x})$.
F. $\frac{1}{2}e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(2e^{-x} + e^{-3x}) + C_3(e^{-2x} - e^{-3x})$.
G. $\frac{1}{2}e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(2e^{-x} + e^{-3x}) + C_3(e^{-2x} - 2e^{-3x})$.
H. $\frac{1}{2}e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(2e^{-x} + e^{-3x}) + C_3(e^{-2x} - 4e^{-3x})$.
I. $\frac{1}{2}e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(2e^{-x} + e^{-3x}) + C_3(2e^{-2x} - e^{-3x})$.
J. $\frac{1}{2}e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(2e^{-x} + e^{-3x}) + C_3(4e^{-2x} - e^{-3x})$.

11. Let

$$L = (D+3)(2D-3) - (3D^2+5).$$

Compute

$$L(\ln x).$$

Evaluate your answer at $x = 1$, and then take its absolute value. What is the result?

A. 0.
B. 1.
C. 2.
D. 3.
E. 4.
F. 5.
G. 6.
H. 7.
I. 8.
J. 9.
K. 10.

12. Earthquakes produce several kinds of waves. The *P-waves* are of particular interest, because they can arrive up to two minutes earlier than the damaging waves of the earthquake and can be used for earthquake early warning systems. If you wanted to use a pendulum to detect 1 Hz P-waves, of the following choices of pendulum length, which one would be the best? That is, for which choice will the amplitude of the response be the largest?

    Warning: The given frequency is in cycles per second. It's not an angular frequency.

    A. 1 cm.
    B. 3 cm.
    C. 10 cm.
    D. 30 cm.
    E. 1 m.
    F. 3 m.
    G. 10 m.
    H. 30 m.
    I. 100 m.
    J. 300 m.
    K. 1 km.

    Note: In reality, P-waves come in a range of frequencies, so you'd want to set up more complex seismometers capable of detecting any frequency in that range.

13. The more destructive waves of an earthquake are surface waves. Like the P-waves in the previous problem, surface waves have a range of frequencies, but for this problem assume that the surface waves have a period of 20 s.

You're designing a skyscraper of mass 2×10^8 kg. Depending on your choices of materials and supports, the sideways wobbling of the skyscraper will act as a mass-spring system with one of the spring constants k below. Which spring constant should you choose so that the skyscraper has the *smallest* response to the surface waves described above?

A. $k = 4 \times 10^6$ N/m.
B. $k = 6 \times 10^6$ N/m.
C. $k = 8 \times 10^6$ N/m.
D. $k = 10 \times 10^6$ N/m.
E. $k = 12 \times 10^6$ N/m.
F. $k = 14 \times 10^6$ N/m.
G. $k = 16 \times 10^6$ N/m.
H. $k = 18 \times 10^6$ N/m.
I. $k = 20 \times 10^6$ N/m.
J. $k = 22 \times 10^6$ N/m.
K. $k = 24 \times 10^6$ N/m.

14. Consider a vertical mass-spring-dashpot system with $m = 1\,\text{kg}$, $c = 5\,\text{N}/(\text{m/s})$, and $k = 50\,\text{N/m}$ subject to an external force of $(50\,\text{N})\cos\omega t$. After the system has settled down to its steady periodic oscillation, you observe that that the mass reaches its maximum point $\frac{1}{8}$-th of a full period after the external force does. What is ω?

A. $1\,\text{rad/s}$.

B. $2\,\text{rad/s}$.

C. $3\,\text{rad/s}$.

D. $4\,\text{rad/s}$.

E. $5\,\text{rad/s}$.

F. $6\,\text{rad/s}$.

G. $7\,\text{rad/s}$.

H. $8\,\text{rad/s}$.

I. $9\,\text{rad/s}$.

J. $10\,\text{rad/s}$.

K. $11\,\text{rad/s}$

15. Use the method of variation of parameters, or any other method you like, to find a particular solution of the following differential equation.

$$y'' + y = 3\csc(2x), \quad 0 < x < \frac{\pi}{2}.$$

16. Find the general (real) solution to the differential equation

$$y^{(4)} + 81y = 4e^{-2x}.$$