Differential Equations (Math 217) Practice Midterm 2

October 25, 2016

- No calculators, notes, or other resources are allowed.
- There are 14 multiple-choice questions, worth 5 points each, and two hand-graded questions, worth 15 points each, for a total of 100 points.
- For the hand-graded questions, please turn in your solution to the two questions to separate piles.
- Write your name and student ID and circle your section on each page of your solutions to the hand-graded questions. There are two questions, so you will do this two times.

The practice midterm and the actual midterm might cover material from *any* of the following:

- Lectures through October 19.
- Chapter 1 and sections 3.1 through 3.3 of the textbook.
- Homework and Webwork through homework and Webwork 8.

- A Bernoulli equation has the form

$$y' + P(x)y = Q(x)y^n,$$

where n is a real number. To solve a Bernoulli equation, use the substitution $v = y^{1-n}$.

- The hyperbolic trigonometric functions are defined by

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}),$$
$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}).$$

- You can compute other logarithms using $\ln(mn) = \ln m + \ln n$ or by interpolating between two known values.

$$\ln 2 \approx 0.69,$$
$$\ln 3 \approx 1.10,$$
$$\ln 5 \approx 1.61,$$
$$\ln 7 \approx 1.95,$$
$$\ln 11 \approx 2.40.$$

- Given an equation of the form

$$M\,dx + N\,dy = 0,$$

let

$$D = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x},$$

If D/N is a function of x only, then

$$e^{\int \frac{D}{N}\,dx}$$

is an integrating factor.

If D/M is a function of y only, then

$$e^{-\int \frac{D}{M}\,dy}$$

is an integrating factor.

One of the following equations is a homogeneous first-order differential equation, one is a Bernoulli equation, and one can be solved with another substitution. Find the corresponding one-parameter families of solutions.

1. $(3x^4 + y^3)\,dx - 3xy^2\,dy = 0.$
2. $(4x^2 + \sin^2 y)\,dx - 2x \sin y \cos y\,dy = 0.$
3. $y(y - x)\,dx + x(x + y)\,dy = 0.$

A. $x^{-1}\sin^2 y - 4x = C.$

B. $y^{-3}e^x - \ln|x| = C.$

C. $\frac{y}{x^2+y^2} = C.$

D. $e^y + x^2e^x = C.$

E. $x \ln|y| - 2e^y = C.$

F. $x^{-1}y^3 - x^3 = C.$

G. $x^2 + xy + y^2 = C.$

H. $y^3e^x - x = C.$

I. $\ln|xy| - xy^{-1} = C.$

J. $x^{-2}e^y - e^x = C.$

K. $x^{-1}y^2 + \ln|x| = C.$

Two of the following equations are exact. Solve those two equations. One of the solutions will be in the left column, and one will be in the right column. The solution in the left column is the answer to question 4, and the solution in the right column is the answer to question 5.

I. $(2x + \ln x)\,dx + \left(\frac{x}{y} + 2y\right)\,dy = 0.$

II. $(3x^2y^3 + y^4)\,dx + (3x^3y^2 + y^4 + 4xy^3)\,dy = 0.$

III. $\left(x^3 + \frac{y}{x}\right)\,dx + (y^2 + \ln x)\,dy = 0.$

IV. $(2xy^2 + 2x)\,dx + (2x^2y + 4x)\,dy = 0.$

V. $(3x^2y + 2x)\,dx + (3xy^2 + 3x^2 + 2y)\,dy = 0.$

4.

A. $2x^3 + y^2 + 6x\ln x = C.$

B. $2x + 2y + xy^{-1} = C.$

C. $x^2y^{-1} - y + 4y\ln x.$

D. $x^2 + y^2 + xy^{-1} = C.$

E. $x^3 + y + \ln x = C.$

F. $x^3 - y^2 = C.$

G. $2x^2 - 5y - 10\ln x = C.$

H. $4x^3 - y^2 - 8x\ln x = C.$

I. $3x^4 + 4y^3 + 12y\ln x = C.$

J. $x^2 + y^2 + x\ln x = C.$

K. $x^2y^{-1} - y^2 = C.$

5.

A. $2x^2 + x^2y^2 + y^2 = C.$

B. $x^3y + x^2 + 2xy^3 + y^2 = C.$

C. $x^3 + 3x^2y^2 + 3xy = C.$

D. $x^2 + 2xy + 2y^2 = C.$

E. $x^4 + 3x^2y^2 + 3y^3 = C.$

F. $x^2 + 4x^2y^2 + 2y^2 = C.$

G. $x^3 + 3x^2y + 2y^3 = C.$

H. $x^3y + xy^2 + 2y = C.$

I. $x^3 + x^2y^2 + 2xy = C.$

J. $x^3y + 2x^2 + xy^3 + y^2 = C.$

K. $5x^3y^3 + 5xy^4 + y^5 = C.$

6. Find a two-parameter family of solutions to the differential equation

$$y'' = 2y(y')^3.$$

A. $y = Ax^2 + B$.
B. $x^{-3} + 2y = Ax + B$.
C. $y^{-2} = Ax + B$.
D. $y^3 + 3x = Ay + B$.
E. $y = 2x^2 + A \ln x + B$.
F. $Ay^2 = 1 + (Ax + B)^2$.
G. $x = Ay^3 + B$.
H. $xy^{-1} = A \ln y + B$.
I. $y^{-3} = Ay + B$.
J. $y = \ln x + Ax^{-1} + B$.
K. $y^3 = A + Be^x$..

7. You have a brilliant idea for solving the problem of ice-cold drinks getting watered down as the ice melts: sugary ice cubes made by freezing sugar water. You are drinking an ice-cold beverage containing 500 mL of liquid, along with lots of your sugar ice cubes. You are engaged in a lively conversation and are thus drinking your drink slowly, at a rate of 20 mL/min. Coincidentally, the ice is also melting at a rate of 20 mL/min, keeping the amount of liquid the same. Initially, the concentration of sugar in the beverage is 0.08 g/mL. Unbeknownst to you, you put way too much sugar in the ice. The concentration of sugar in the melting ice is 0.20 g/mL. You'll notice your beverage becoming sweeter when the concentration of sugar of your beverage rises above 0.10 g/mL. How long after you start drinking the beverage will the concentration of sugar rise above 0.10 g/mL?

   A. Between one and two minutes.
   B. Between two and three minutes.
   C. Between three and four minutes.
   D. Between four and five minutes.
   E. Between five and six minutes.
   F. Between six and seven minutes.
   G. Between seven and eight minutes.
   H. Between eight and nine minutes.
   I. Between nine and ten minutes.
   J. Between ten and eleven minutes.
   K. Between eleven and twelve minutes.

8. Which of the following is an exact equation with general solution

$$y = \pm\sqrt{x + Cx^{-1}}?$$

A. $(2x - y)\,dx + 6y\,dy = 0.$
B. $(2xy - 6x^3)\,dx + x^2\,dy = 0.$
C. $(4x - y)\,dx + (8y - x)\,dy = 0.$
D. $(4x + 1)\,dx - 2x^{-2}\,dy = 0.$
E. $2x\,dx + (3x + 4y)\,dy = 0.$
F. $(xy^2 + x^2)\,dx + (x^2y + y^3)\,dy = 0.$
G. $(x^2 + 2y^2)\,dx + (4xy + 6y^2)\,dy = 0.$
H. $y\,dx + 2(x - x^{-3})\,dy = 0.$
I. $(3x + 3y)\,dx + (2x + 3y)\,dy = 0.$
J. $(y^2 - 2x)\,dx + 2xy\,dy = 0.$
K. $(2xy^2 + 3x^2)\,dx + x^2y\,dy = 0.$

9. Find the general solution to the equation

$$\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0.$$

A. $y = (C_1 + C_2x)e^{2x} + C_3\cos(2x) + C_4\sin(2x)$.
B. $y = C_1\cos(2x) + C_2\sin(2x) + x(C_3\cos(2x) + C_4\sin(2x))$.
C. $y = C_1e^{2x} + C_2xe^{2x} + C_3e^{2x}\cos(2x) + C_4e^{2x}\sin(2x)$.
D. $y = (C_1 + C_2x)(C_3\cosh(2x) + C_4\sinh(2x))$.
E. $y = C_1 + C_2x + e^{2x}(C_3\cos(2x) + C_4\sin(2x))$.
F. $y = (C_1 + C_2x)e^{2x}(C_3\cos(2x) + C_4\sin(2x))$.
G. $y = (C_1 + C_2x)(C_3\cos(2x) + C_4\sin(2x))$
H. $y = (C_1 + C_2x)e^{2x}\cos(2x) + (C_3 + C_4x)e^{2x}\sin(2x)$.
I. $y = C_1\cosh(2x) + C_2\sinh(2x) + C_3\cos(2x) + C_4\sin(2x)$.
J. $y = C_1 + C_2x + C_3\cos(2x) + C_4\sin(2x)$.
K. $y = C_1\cosh(2x) + C_2\sinh(2x) + C_3x\cosh(2x) + C_4x\sinh(2x)$.

10. The function $y = \cosh 2x$ is a solution to the differential equation

$$y'' + 5y' + 6y = 10e^{2x}.$$

Which of the following is also a solution to this differential equation?

A. $\frac{1}{2}e^{2x}$.
B. $\frac{1}{2}e^{-5x}$.
C. $\frac{1}{2}\sinh 2x$.
D. $\frac{1}{2}\sin 2x$.
E. $\frac{1}{2}$.
F. $\frac{1}{2}e^{-3x}$.
G. $\frac{1}{2}\cos 2x$.
H. $\frac{1}{2}\cosh 2x$.
I. $\frac{1}{2}e^{-2x}$.
J. $\frac{1}{2}e^{3x}$.
K. $\frac{1}{2}e^{5x}$.

11. Let

$$L = (D+1)(D+2)(2D+1) - D^4 + 20.$$

Compute

$$L\left(e^{-2x}\right).$$

A. $-5e^{-2x}$.
B. $-4e^{-2x}$.
C. $-3e^{-2x}$.
D. $-2e^{-2x}$.
E. $-e^{-2x}$.
F. 0.
G. e^{-2x}.
H. $2e^{-2x}$.
I. $3e^{-2x}$.
J. $4e^{-2x}$.
K. $5e^{-2x}$.

12. Two of the solutions of a linear homogeneous differential equation with constant coefficients are

$$y_1 = -3x^2 e^{-3x}, \text{ and } \qquad y_2 = 3\cos 3x.$$

What is the minimum possible order of the differential equation?

A. One.
B. Two.
C. Three.
D. Four.
E. Five.
F. Six.
G. Seven.
H. Eight.
I. Nine.
J. Ten.
K. Eleven.

13. Each of the choices presents three functions y_1, y_2, and y_3, which are solutions to a third-order homogeneous linear equation of the form

$$y^{(3)} + p_1(x)y'' + p_2(x)y' + p_3(x)y = 0,$$

where p_1, p_2, p_3 are continuous near $x = 0$. For exactly one of the choices, the solution to the initial value problem with the differential equation above and the initial conditions

$$y(0) = 2, \qquad y'(0) = 1, \qquad y''(0) = -2.$$

can be written in the form

$$y = C_1y_1 + C_2y_2 + C_3y_3$$

for constants C_1, C_2, and C_3. Which choice is it?

A. $y_1 = 2e^x$, $y_2 = e^{2x}$, $y_3 = e^{x+2}$.
B. $y_1 = 2e^{2x}$, $y_2 = -2e^{2x}$, $y_3 = 2e^{-2x}$.
C. $y_1 = \cos^2 x$, $y_2 = \cos 2x$, $y_3 = 2$.
D. $y_1 = e^{2x}$, $y_2 = e^{-2x}$, $y_3 = 2e^x$.
E. $y_1 = \cosh 2x$, $y_2 = \sinh 2x$, $y_3 = -2e^{-2x}$.
F. $y_1 = \cos^2 x$, $y_2 = \sin^2 x$, $y_3 = -2$.

14. Let y be the solution to the initial value problem

$$y^{(4)} + 6y'' + 9y = 0 \qquad \begin{aligned} y(1) &= -2, \\ y'(1) &= 2, \\ y''(1) &= 3, \\ y^{(3)}(1) &= -3. \end{aligned}$$

Compute $y^{(5)}(1)$. Hint: Take the derivative of the differential equation.

A. -5.

B. -4.

C. -3.

D. -2.

E. -1.

F. 0.

G. 1.

H. 2.

I. 3.

J. 4.

K. 5.

15. Write your answer to this question on the free-response page.

Consider the general first-order differential equation

$$M dx + N dy = 0, \quad (\dagger)$$

where M and N are functions of x and y. In the homework, we determined how to find an integrating factor that depends only on x or an integrating factor that depends only on y. In this problem, we will find a condition under which ($\dagger$) has an integrating factor which only depends on xy.

(a) (4 points) We're looking for a function $I(z)$ such that when we set $z = xy$ and multiply the above equation by $I(xy)$, we get an exact equation. Multiplying by $I(xy)$, we get

$$M \cdot I(xy)\, dx + N \cdot I(xy)\, dy = 0.$$

Write down the criterion for exactness for this equation in terms of M and N, their partial derivatives, I, and I'. Be careful to apply the chain rule correctly when taking the partial derivatives of $I(xy)$.

(b) (4 points) Rearrange your equation from the previous part so that the terms involving M and N and their derivatives are on the left-hand side and the terms inolving I and I' are on the right-hand side. The right-hand side is a function that only depends on xy. Conclude that the left-hand side must also depend only on xy.

(c) (3 points) To simplify your work, let the left-hand side be $P(z)$, where $z = xy$. Solve a differential equation to find I in terms of P.

(d) (4 points) Use the previous parts, or any other methods you want, to solve the following differential equation.

$$y dx + \left(x - 2x^2y^3\right) dy = 0. \quad (\ddagger)$$

15. The text of this question is on the last page of the multiple-choice section.

16. Make sure to show your work and justify your answers.

(a) (10 points) A tank initially contains 60 gal of water with initial salt concentration 4 lbs/gal. Brine flows into the tank at a rate of 3 gal/min. The concentration of salt of the incoming brine varies with time. At t minutes, the concentration is $2 + \cos t$ pounds per gallon. The well-mixed water in the tank flows out of the tank at a rate of 1 gal/min.

Write down a differential equation with independent variable t and one dependent variable for the amount of salt in the tank.

(b) (5 points) Write down a third-order homogeneous linear differential equation with constant coefficients with independent variable x and dependent variable y, such that as x approaches *positive* infinity, every solution approaches a sinusoid.