Differential Equations (Math 217) Practice Midterm 1

September 20, 2016

- No calculators, notes, or other resources are allowed.
- There are 14 multiple-choice questions, worth 5 points each, and two hand-graded questions, worth 15 points each, for a total of 100 points.
- For the hand-graded questions, please turn in your solution to question 15 to one pile and your solution to question 16 to a separate pile.
- Write your name and student ID and circle your section on each page of your solutions to the hand-graded questions. The two questions span three pages, so you will do this three times.

The format and the content of the midterm will be similar to this practice midterm, and might cover material from *any* of the following:

- Lectures through September 14.
- Sections 1.1 through 1.4, and the first part of section 1.5 (until "A Closer Look").
- Homework and Webwork 1–3.
  - Webwork 4 is not on this list, but some Webwork 4 problems may be helpful practice on the linear equation material included in the first two bullet points.

Match the following differential equations to their solutions.

1. $x^2y'' + 5xy' + 4y = 0$.
2. $xy' = y + 2\sqrt{xy}$.
3. $xe^y y' = 2(e^y + x^3e^{2x})$.
4. $xy' + 6y = 3xy^{4/3}$.
5. $x^2y'' + xy' - y = \ln x$.

   A. $y = (x + Cx^2)^{-3}$ or $y = 0$.
   B. $y = C_1x + C_2\frac{1}{x} - \ln x$.
   C. $y = \ln(Cx^2 + x^2e^{2x})$.
   D. $y = x(C + \ln|x|)^2$ or $y = 0$.
   E. $y = C_1\frac{1}{x^2} + C_2\frac{\ln x}{x^2}$.

What are the solutions to the following differential equations?

6. $y' + 3y = 2xe^{-3x}$.
7. $y' = 4x^3y - y$.
8. $xy' + 2y = 3x$.

A. $y = \frac{1}{2}(1 + Ce^{2x-4})$.
B. $y = C(2 - x)^4$.
C. $y = (2x^{1/2} + C)^2$.
D. $y = Ce^{x-\frac{1}{2}x^2}$.
E. $y = Ce^{x^4-x}$.
F. $y = x + \frac{C}{x^2}$.
G. $y = (x^{2/3} + C)^2$.
H. $y = (x + C)e^{x^2}$.
I. $y = 2x^{1/2} + Cx^{-1/2}$.
J. $y = e^{-3x}(x^2 + C)$.
K. $y = x^2(C + \ln|x|)$.

9. You're writing a science fiction novel in which your characters take a trip to Saturn. You feel that "warp drives" are silly and would like to keep things a bit more realistic, so your characters' spaceship accelerates at $9.8\,\mathrm{m/s^2}$, conveniently simulating Earth's gravity. Much more acceleration than that would squish your characters to the back of the spaceship. You're ignoring the amount of fuel it would take to accomplish this feat, since it's science *fiction*.

   When the spaceship gets halfway to Saturn, it starts decelerating, again at $9.8\,\mathrm{m/s^2}$, conveniently coming to a stop right at Saturn so that your characters can snap pictures of Saturn's rings and the plot of your story can happen. The distance between the Earth and Saturn ranges between 1.2 and 1.7 billion kilometers, depending on where the two planets are in their orbits. For this problem, go with a distance of 1.6 billion kilometers, approximate the acceleration as $10\,\mathrm{m/s^2}$, and, if needed, approximate $\sqrt{10} \approx 3$. How long does it take your characters to get to Saturn?

   A. 8 seconds.
   B. 80 seconds.
   C. 800 seconds (13 minutes).
   D. 8000 seconds (2 hours).
   E. 80 000 seconds (20 hours).
   F. 800 000 seconds (9 days).
   G. 8 000 000 seconds (90 days).
   H. 80 000 000 seconds (2.5 years).
   I. 800 000 000 seconds (25 years).
   J. 8 000 000 000 seconds (250 years).
   K. 80 000 000 000 seconds (2500 years).

The next two questions are about a torsion balance, also known as a torsion pendulum, shown below.

The rod is suspended from a thin wire. As the rod rotates in the horizontal plane, the wire twists and gently pulls the rod back. While springs are good at measuring medium and big forces, torsion balances let you measure tiny forces, most notably used in 1798 to measure the mass of the Earth.

The *moment of inertia* I of a rigid body is a constant that represents how much it resists being spun, and, in this case, depends on the masses of the weights attached to the rod are and their distance from the center. A *torque* τ is a force acting at a distance away from the center. Like with wrenches, the further from the center, the bigger the torque.

Let θ be the angle the rod rotates from its equilibrium position. The torque on a torsion balance satisfies an angular analog of Hooke's law

$$\tau = -k\theta,$$

for some constant k.

The *angular velocity* ω of the rod is simply the rate of change of θ:

$$\omega = \frac{d\theta}{dt}.$$

Meanwhile, the *angular momentum* of the rod has the formula

$$L = I\omega.$$

Finally, torque causes a change in angular momentum via the formula

$$\tau = \frac{dL}{dt}.$$

10. Substitute to obtain a second-order differential equation in terms of θ, t, and the constants I and k, but not ω, L, or τ. Then write it in a form that looks similar to the equation of a spring from class. Which of the following is a solution to this equation?

    A. $\theta = \cos(Ikt)$.
    B. $\theta = \cos\left(\frac{I}{k}t\right)$.
    C. $\theta = \cos\left(\frac{k}{I}t\right)$.
    D. $\theta = \cos\left(\frac{1}{Ik}t\right)$.
    E. $\theta = \cos\left(\sqrt{Ik}t\right)$.
    F. $\theta = \cos\left(\sqrt{\frac{I}{k}}t\right)$.
    G. $\theta = \cos\left(\sqrt{\frac{k}{I}}t\right)$.
    H. $\theta = \cos\left(\sqrt{\frac{1}{Ik}}t\right)$.

11. One can determine the moment of intertia I from the masses and sizes of the weights and the rod. Determining the constant k is trickier. With a spring, one can determine k by hanging a known mass off of the spring and seeing how much it stretches. However, because the torsion balance is so sensitive, any reasonably-sized torque will cause it to spin out of control. Instead, the way to determine k is to give the rod a small push, and then record the period of the resulting oscillation. For Cavendish's experiment to measure the mass of the Earth, the period was about 20 minutes, or 1200 seconds. Meanwhile, the rod and attached weights had a moment of inertia of about $8\,\mathrm{kg\,m^2}$. Determine k, approximately. If needed, approximate $\pi \approx 3$ and $\sqrt{10} \approx 3$.

    A. $2 \times 10^{-5}\,\mathrm{kg\,m^2/s^2}$.
    B. $2 \times 10^{-4}\,\mathrm{kg\,m^2/s^2}$.
    C. $2 \times 10^{-3}\,\mathrm{kg\,m^2/s^2}$.
    D. $0.02\,\mathrm{kg\,m^2/s^2}$.
    E. $0.2\,\mathrm{kg\,m^2/s^2}$.
    F. $2\,\mathrm{kg\,m^2/s^2}$.
    G. $20\,\mathrm{kg\,m^2/s^2}$.
    H. $200\,\mathrm{kg\,m^2/s^2}$.
    I. $2 \times 10^{3}\,\mathrm{kg\,m^2/s^2}$.
    J. $2 \times 10^{4}\,\mathrm{kg\,m^2/s^2}$.
    K. $2 \times 10^{5}\,\mathrm{kg\,m^2/s^2}$.

12. For exactly one of the following differential equations, there is no solution to the intial value problem with initial condition $y(-1) = 1$. Which differential equation is it?

A. $y' = (x-1)^{6/5}(y-1)^{6/5}$.

B. $y' = (x+1)^{-6/5}(y+1)^{6/5}$.

C. $y' = (x+1)^{2/5}(y+1)^{2/5}$.

D. $y' = (x-1)^{6/5}(y+1)^{-2/5}$.

E. $y' = (x+1)^{2/5}(y-1)^{6/5}$.

13. The terminal velocity of a falling cat is 60 mph, or about 25 m/s for the purposes of this problem. That is, if a cat falls from a large height, in the long run, its velocity will appraoch 25 m/s.

If we ignore air resistance, the downward velocity v of the cat satisfies the differential equation

$$\frac{dv}{dt} = g,$$

where g is the gravitational constant, which for this problem you can approximate as $10\,\text{m/s}^2$. On the other hand, if we take into account air resistance, our model becomes

$$\frac{dv}{dt} = g - kv,$$

for some constant k. From the negative sign, we see that air resistance counteracts gravity, and the effect of air resistance is larger the faster the cat falls, as we would expect.

By rewriting the equation to look like Newton's law of cooling or by any other means, determine k. Approximately how long after the cat falls out the window does it reach 80% of its final speed, or 20 m/s? Note: $\ln 5 \approx 1.6$.

A. Less than half a second.

B. 1 second.

C. 2 seconds.

D. 3 seconds.

E. 4 seconds.

F. 5 seconds.

G. 6 seconds.

H. 7 seconds.

I. 8 seconds.

J. 9 seconds.

K. 10 seconds.

14. Consider the differential equation

$$yy' = -\tfrac{1}{9}x.$$

Find an implicit solution to this differential equation. Using your solution, find the largest domain on which the solution to the initial value problem

$$yy' = -\tfrac{1}{9}x, \qquad\qquad y(4) = -1.$$

is defined.

A. $[0, \infty)$.
B. $[-1, \infty)$.
C. $[-2, \infty)$.
D. $[-3, \infty)$.
E. $[-4, \infty)$.
F. $[-5, \infty)$.
G. $[-5, 5]$.
H. $[-6, \infty)$.
I. $[-6, 6]$.
J. $[-7, \infty)$.
K. $[-7, 7]$.

15. (15 points) Find a general solution for the following differential equation.

$$\frac{dy}{dx} = 2xy^2 - x^3y^2.$$

Show your work.

16. In class, we talked about Newton's law of cooling. However, in real life, the ambient temperature of the environment varies with time. Let T be the temperature inside an unheated shed in degrees Celsius, and let t be time in hours, with $t = 0$ representing midnight. Our model is

$$\frac{dT}{dt} = 0.2\left(20 - 5\sin\left(2\pi\tfrac{t}{24}\right) - T\right).$$

   (a) (1 point) Which expression in the model represents the ambient temperature of the environment?

   (b) (1 point) The ambient temperature of the environment is periodic. What is its period? Include units.

   (c) (1 point) Name a real-life factor that could be modeled using such a periodic ambient temperature.

   (d) (4 points) The slope field for this differential equation is on the next page. Sketch five solution curves that represent the temperature of the shed if at $t = 0$ its initial temperature is each of 0 °C, 10 °C, 20 °C, 25 °C, and 30 °C.

   (e) It looks like all of the solutions eventually approach the same periodic solution, which is the *steady state solution* and describes how the temperature of the shed will behave in the long run regardless of the initial temperature. Fill in the blanks.

      i. (2 points) In the long run, the temperature of the shed will vary between approximately ______________ °C and approximately ______________ °C.

      ii. (2 points) Meanwhile, the temperature of the environment varies between ______________ °C and ______________ °C.

      iii. (2 points) In the long run, the temperature of the shed will be at its lowest at around ______________ and at its highest at around ______________. (Enter a time of day, accurate to within an hour.)

      iv. (2 points) Meanwhile, the temperature of the environment is at its lowest at ______________ and highest at ______________. (Enter an exact time of day.)

Math 217 **Name:** ____________________

Practice Midterm 1 **Student ID:** ____________________

September 20, 2016 **Circle one: 9am 11am**