Differential Equations (Math 217) Midterm 3

November 15, 2016

- No calculators, notes, or other resources are allowed.
- There are 14 multiple-choice questions, worth 5 points each, and two hand-graded questions, worth 15 points each, for a total of 100 points.
- For the hand-graded questions, please turn in your solution to the two questions to separate piles.
- Write your name and student ID and circle your section on each page of your solutions to the hand-graded questions. There are two questions, so you will do this two times.

- The hyperbolic trigonometric functions are defined by

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}),$$
$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}).$$

- You can compute other logarithms using $\ln(mn) = \ln m + \ln n$ or by interpolating between two known values.

$$\ln 2 \approx 0.69,$$
$$\ln 3 \approx 1.10,$$
$$\ln 5 \approx 1.61,$$
$$\ln 7 \approx 1.95,$$
$$\ln 11 \approx 2.40.$$

- Other approximations:
  - $\pi \approx 3$.
  - $g \approx 10\,\mathrm{m/s^2}$.

- If n is positive, to determine whether $n < \sqrt{x}$, determine whether $n^2 < x$.

- The method of variation of parameters involves the system of equations

$$u_1' y_1 + u_2' y_2 = 0,$$
$$u_1' y_1' + u_2' y_2' = f(x).$$

1. Determine the amplitude of the solution to the initial value problem

$$y'' + 9y = 0, \qquad y(0) = 5, \qquad y'(0) = -21.$$

Decide whether the amplitude is:

A. Between zero and one.
B. Between one and two.
C. Between two and three.
D. Between three and four.
E. Between four and five.
F. Between five and six.
G. Between six and seven.
H. Between seven and eight.
I. Between eight and nine.
J. Between nine and ten.
K. Between ten and eleven.

2. Find a solution y_p to the differential equation

$$y^{(3)} - 3y'' - 2y' + 2y = 12e^{-2x}$$

that is either the product of a polynomial and an exponential, or the product of a polynomial, an exponential, and a sinusoid, keeping in mind that a constant polynomial counts as a polynomial. Evaluate $y_p'(0)$.

A. $\frac{3}{2}$.
B. $\frac{4}{3}$.
C. 24.
D. 12.
E. $\frac{24}{19}$.
F. $-\frac{12}{5}$.
G. $\frac{12}{7}$.
H. $\frac{8}{5}$.
I. $\frac{6}{5}$.
J. $-\frac{6}{5}$.
K. $-\frac{8}{5}$.

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4. A car has a mass of 2000 kg. The shock absorbers in the car act as a dashpot acting with a force of 20 000 N/(m/s). Meanwhile, the suspension acts as a spring with spring constant k. You've got a couple choices for the suspension:

- Suspension I: $k = 10\,000$ N/m.
- Suspension II: $k = 20\,000$ N/m.
- Suspension III: $k = 30\,000$ N/m.
- Suspension IV: $k = 40\,000$ N/m.
- Suspension V: $k = 50\,000$ N/m.

You'd like a situation in which, after hitting a bump, the car crosses its equilibrium position at most once. Of these choices, with which suspension systems will that happen?

A. I.

B. I and II.

C. I, II, and III.

D. I, II, III, and IV.

E. I, II, III, IV, and V.

F. II, III, IV, and V.

G. III IV, and V.

H. IV and V.

I. V.

J. None of them.

5. Consider the periodic motion

$$x(t) = -3\sqrt{2}\cos(4t) - 3\sqrt{2}\sin(4t).$$

At what time does the second minimum of x occur, starting at $t = 0$?

A. $t = \frac{1}{16}\pi$.
B. $t = \frac{3}{16}\pi$.
C. $t = \frac{5}{16}\pi$.
D. $t = \frac{7}{16}\pi$.
E. $t = \frac{9}{16}\pi$.
F. $t = \frac{11}{16}\pi$.
G. $t = \frac{13}{16}\pi$.
H. $t = \frac{15}{16}\pi$.
I. $t = \frac{17}{16}\pi$.
J. $t = \frac{19}{16}\pi$.
K. $t = \frac{21}{16}\pi$.

6. Consider a mass-spring-dashpot system with $m = 3\,\mathrm{kg}$, $c = 2\,\mathrm{N}/(\mathrm{m}/\mathrm{s})$, and $k = 100\,\mathrm{N}/\mathrm{m}$ under the influence of an external force $F(t) = (6\,\mathrm{N})\cos(10t)$, where t is in seconds. What is the *approximate* amplitude of the steady periodic response of this mass-spring-dashpot system to the external force?

   A. $0.01\,\mathrm{m}$.

   B. $0.03\,\mathrm{m}$.

   C. $0.06\,\mathrm{m}$.

   D. $0.1\,\mathrm{m}$.

   E. $0.3\,\mathrm{m}$.

   F. $0.6\,\mathrm{m}$.

   G. $1\,\mathrm{m}$.

   H. $3\,\mathrm{m}$.

   I. $6\,\mathrm{m}$.

   J. $10\,\mathrm{m}$.

   K. $30\,\mathrm{m}$.

7. The Gateway Arch is 192 m tall, and is conveniently the right shape for installing a swing set.

We can see from the picture that swing is attached below the top of the arch. Assume that the chains are attached 160 m above the ground. Ignoring air resistance, *approximately* how long does it take the kid to complete one swing back and forth (a complete period)?

A. 0.01 seconds.
B. 0.02 seconds.
C. 0.05 seconds.
D. 0.25 seconds.
E. 1 second.
F. 5 seconds.
G. 15 seconds.
H. 25 seconds.
I. 50 seconds.
J. 100 seconds.
K. 200 seconds.

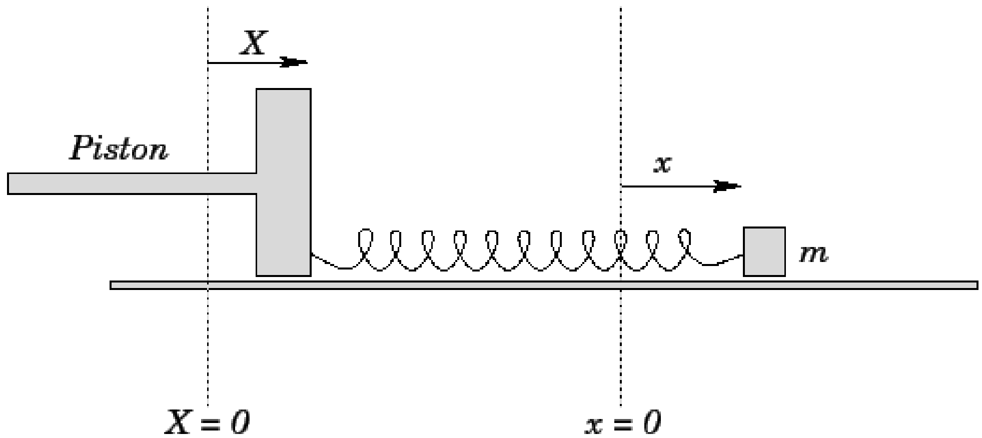
8. This question also concerns a giant swing set, but this time we will not ignore air resistance. For the average human travelling at 1 m/s, the air provides a force that decelerates them at a rate of approximately $0.2\,\mathrm{m/s^2}$. This force scales proportionally to the human's speed.

   With a tall enough swingset, one expects that the kid will go so fast that the force of the oncoming air will bring the kid to a stop right at the bottom of the arch. Find the minimum height of a swingset on which it is impossible to swing, that is, where if the kid were to be released from rest, he would travel to the bottom of the arch, but stop there without crossing it.

   Note: Be mindful of the fact that speed is the size of the rate of change of displacement and thus has units m/s.

   A. 10 m.
   B. 20 m.
   C. 50 m.
   D. 100 m.
   E. 200 m.
   F. 500 m.
   G. 1000 m.
   H. 2000 m.
   I. 5000 m.
   J. 10 000 m.
   K. 20 000 m.

9. Consider the following forced harmonic oscillator with positive but negligible damping.



When the piston does not move, if we set the mass m in motion, it oscillates at a frequency of 10 Hz. When the piston oscillates at a frequency of 12 Hz with an amplitude of 22 cm, we observe that, eventually, the motion of the mass m settles down to a sinusoid also of frequency 12 Hz. What is the amplitude of this sinusoid?

A. 10 cm.
B. 20 cm.
C. 30 cm.
D. 40 cm.
E. 50 cm.
F. 60 cm.
G. 70 cm.
H. 80 cm.
I. 90 cm.
J. 100 cm.
K. 110 cm.

10. The answer choices below represent families of functions. For one of the choices, there is a solution to the differential equation

$$y''' + 6y'' + 11y' + 6y = 24e^x$$

that is *not* a member of the family. Which choice is it?

A. $e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(e^{-x} - 2e^{-3x}) + C_3(e^{-2x} + e^{-3x})$.
B. $e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(e^{-x} - 2e^{-3x}) + C_3(e^{-2x} + 2e^{-3x})$.
C. $e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(e^{-x} - 2e^{-3x}) + C_3(e^{-2x} + 4e^{-3x})$.
D. $e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(e^{-x} - 2e^{-3x}) + C_3(2e^{-2x} + e^{-3x})$.
E. $e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(e^{-x} - 2e^{-3x}) + C_3(4e^{-2x} + e^{-3x})$.
F. $e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(e^{-x} - 2e^{-3x}) + C_3(e^{-2x} - e^{-3x})$.
G. $e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(e^{-x} - 2e^{-3x}) + C_3(e^{-2x} - 2e^{-3x})$.
H. $e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(e^{-x} - 2e^{-3x}) + C_3(e^{-2x} - 4e^{-3x})$.
I. $e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(e^{-x} - 2e^{-3x}) + C_3(2e^{-2x} - e^{-3x})$.
J. $e^x + C_1(e^{-x} + 2e^{-2x}) + C_2(e^{-x} - 2e^{-3x}) + C_3(4e^{-2x} - e^{-3x})$.

11. Let

$$L = (D+2)(2D+3) - 2D^2 - 4.$$

Compute

$$L(\ln x).$$

Evaluate your answer at $x = 1$, and then take its absolute value. What is the result?

A. 0.

B. 1.

C. 2.

D. 3.

E. 4.

F. 5.

G. 6.

H. 7.

I. 8.

J. 9.

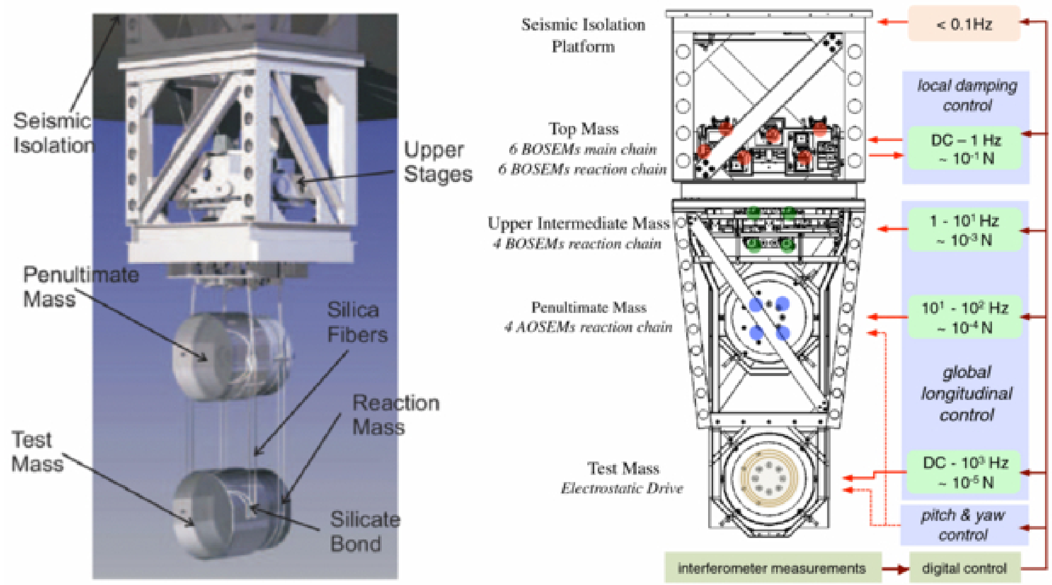
K. 10.

12. The Laser Interferometer Gravitational-Wave Observatory (LIGO) detects gravitational waves by measuring tiny changes in the distance between two mirrors 4 km apart. To do so, the mirror needs to be isolated from all seismic noise, as the observatory is sensitive enough to detect vibrations from traffic on nearby roads. The 40 kg mirror hangs from silica fibers of length 60 cm, which are attached to a seismic isolation platform.

    Treating the mirror as a pendulum, which of the following angular frequencies does the seismic isolation platform need to be most careful about? That is, of the following choices of ω, for which one will a force $F(t) = \cos \omega t$ cause the largest steady periodic response of the pendulum?

    A. $\omega = 1\,\text{rad/s}$.
    B. $\omega = 2\,\text{rad/s}$.
    C. $\omega = 3\,\text{rad/s}$.
    D. $\omega = 4\,\text{rad/s}$.
    E. $\omega = 5\,\text{rad/s}$.
    F. $\omega = 6\,\text{rad/s}$.
    G. $\omega = 7\,\text{rad/s}$.
    H. $\omega = 8\,\text{rad/s}$.
    I. $\omega = 9\,\text{rad/s}$.
    J. $\omega = 10\,\text{rad/s}$.
    K. $\omega = 11\,\text{rad/s}$.

    Note: In reality, the mirror is not attached to a seismic isolation platform. Instead, it's attached to the bottom of another pendulum, which in turn is attached to yet another pendulum, which is attached to a fourth pendulum. That pendulum is attached to the seismic isolation platform, which is actually several seismic isolation platforms stacked on top of each other, each with a specialized control system designed to cancel out a particular range of frequencies of seismic vibrations (think noise-cancelling headphones).



tl;dr detecting changes in distance smaller than the size of a proton takes a lot of work.

13. Machinery in the basement of Cupples I, presumably related to the ventillation system, vibrates at an angular frequency of 30 rad/s (regular frequency 4.8 Hz). This seems to be near the resonant frequency of the desk in my office, and consequently it wobbles at an angular frequency of 30 rad/s, making an annoying noise. Model my desk's resistance to wobbling as a spring with spring constant $k = 40\,000\,\mathrm{N/m}$ and positive but negligible damping. By removing or adding items to my desk, I can make the total mass of my desk any of the following values. Of these values, which one should I choose to make the wobbling of my desk as *small* as possible?

    A. $m = 40\,\mathrm{kg}$.
    B. $m = 42\,\mathrm{kg}$.
    C. $m = 44\,\mathrm{kg}$.
    D. $m = 46\,\mathrm{kg}$.
    E. $m = 48\,\mathrm{kg}$.
    F. $m = 50\,\mathrm{kg}$.
    G. $m = 52\,\mathrm{kg}$.
    H. $m = 54\,\mathrm{kg}$.
    I. $m = 56\,\mathrm{kg}$.
    J. $m = 58\,\mathrm{kg}$.
    K. $m = 60\,\mathrm{kg}$.

14. Consider a vertical mass-spring-dashpot system with $m = 1\,\mathrm{kg}$, $c = 4\,\mathrm{N/(m/s)}$, and $k = 100\,\mathrm{N/m}$ subject to an external force of $(100\,\mathrm{N})\cos\omega t$. After the system has settled down to its steady periodic oscillation, you observe that that the mass is travelling upwards whenever the external force is positive (upwards), and the mass is travelling downwards whenever the external force is negative (downwards). What is the amplitude of the steady periodic oscillation?

A. 2.5 cm.

B. 5.0 cm.

C. 7.5 cm.

D. 10 cm.

E. 25 cm.

F. 50 cm.

G. 75 cm.

H. 1.0 m.

I. 2.5 m.

J. 5.0 m.

K. 7.5 m

15. Use the method of variation of parameters, or any other method you like, to find a particular solution of the following differential equation.

$$y'' - 2y' + y = \frac{e^x}{1 + x^2}.$$

16. Find the general (real) solution to the differential equation

$$y^{(6)} = -64y + 7e^{-x}.$$