Differential Equations (Math 217) Midterm 2

October 25, 2016

- No calculators, notes, or other resources are allowed.
- There are 14 multiple-choice questions, worth 5 points each, and two hand-graded questions, worth 15 points each, for a total of 100 points.
- For the hand-graded questions, please turn in your solution to the two questions to separate piles.
- Write your name and student ID and circle your section on each page of your solutions to the hand-graded questions. There are two questions, so you will do this two times.

- A Bernoulli equation has the form
$$y' + P(x)y = Q(x)y^n,$$
where n is a real number. To solve a Bernoulli equation, use the substitution $v = y^{1-n}$.

- The hyperbolic trigonometric functions are defined by
$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}),$$
$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}).$$

- You can compute other logarithms using $\ln(mn) = \ln m + \ln n$ or by interpolating between two known values.
$$\ln 2 \approx 0.69,$$
$$\ln 3 \approx 1.10,$$
$$\ln 5 \approx 1.61,$$
$$\ln 7 \approx 1.95,$$
$$\ln 11 \approx 2.40.$$

- Given an equation of the form
$$M\,dx + N\,dy = 0,$$
let
$$D = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x},$$
If D/N is a function of x only, then
$$e^{\int \frac{D}{N}\,dx}$$
is an integrating factor.
If D/M is a function of y only, then
$$e^{-\int \frac{D}{M}\,dy}$$
is an integrating factor.

One of the following equations is a homogeneous first-order differential equation, one is a Bernoulli equation, and one can be solved with another substitution. Find the corresponding one-parameter families of solutions.

1. $(2y^3 - e^{-2x})\,dx + 3y^2\,dy = 0$.
2. $2(e^y + x^3e^x)\,dx - xe^y\,dy = 0$.
3. $(x^2 - 2y^2)\,dx + 2xy\,dy = 0$.

A. $x^{-1}\sin^2 y - 2x^2 = C$.
B. $x^2 + 2xy + y^2 = C$.
C. $y^3e^{2x} - x = C$.
D. $x^{-2}e^y - 2e^x = C$.
E. $\frac{2y}{x^2+y^2} = C$.
F. $\ln|xy| - 2xy^{-1} = C$.
G. $x^{-1}y^3 - x^3 = C$.
H. $y^{-3}e^{2x} - \ln|x| = C$.
I. $e^y + 3x^2e^x = C$.
J. $x^{-2}y^2 + \ln|x| = C$.
K. $x\ln|y| + e^y = C$.

Two of the following equations are exact. Solve those two equations. One of the solutions will be in the left column, and one will be in the right column. The solution in the left column is the answer to question 4, and the solution in the right column is the answer to question 5.

I. $(2x + \ln y)\,dx + \left(\frac{x}{y} + 2y\right)\,dy = 0.$

II. $(3x^2y + y^2)\,dx + (3x^3 + y^2 + 4xy)\,dy = 0.$

III. $(3x^2y + 2x)\,dx + (3xy^2 + 3x^2 + 2y)\,dy = 0.$

IV. $\left(\frac{2x}{y} - 2y\right)\,dx + (\ln y + 3x^2)\,dy = 0.$

V. $(2xy^2 + 2x)\,dx + (2x^2y + 4y)\,dy = 0.$

4.

A. $x^2 + y^2 + x\ln y = C.$

B. $x^3 - y^2 - x\ln y = C.$

C. $2x^3 + y^2 + x\ln y = C.$

D. $x^2y^{-1} - y + y\ln y.$

E. $x^3 + y^2 + xy^{-1} = C.$

F. $x^2 - y - \ln y = C.$

G. $x^3 + y + \ln y = C.$

H. $x^2 + y^2 + xy^{-1} = C.$

I. $x^3 - y^2 = C.$

J. $2x + 2y + xy^{-1} = C.$

K. $x^2y^{-1} - y^2 = C.$

5.

A. $x^3y + x^2 + xy^3 + y^2 = C.$

B. $2x^2 + 3x^2y^2 + y^2 = C.$

C. $x^2 + 2xy + y^2 = C.$

D. $5x^3y^3 + 5xy^4 + y^5 = C.$

E. $x^3y + 2x^2 + 2xy^3 + y^2 = C.$

F. $x^3 + 3x^2y^2 + 2xy = C.$

G. $x^4 + \frac{3}{2}x^2y^2 + \frac{1}{3}y^3 = C.$

H. $x^3 + 2x^2y + 2y^3 = C.$

I. $x^2 + x^2y^2 + 2y^2 = C.$

J. $x^3y + 2xy^2 + \frac{1}{2}y = C.$

K. $x^3 + \frac{3}{2}x^2y^2 + 2xy = C.$

6. Find a two-parameter family of solutions to the differential equation

$$yy'' = 4(y')^2.$$

A. $x^{-3} + 3y = Ax + B.$
B. $y^{-2} = Ay + B.$
C. $y = Ax^2 + B.$
D. $y^3 + 3x = Ay + B.$
E. $y = x^2 + A\ln x + B.$
F. $xy^{-1} = A\ln y + B.$
G. $Ay^2 = 1 + (Ax + B)^2.$
H. $y^{-3} = Ax + B.$
I. $y^2 = A + Be^x.$
J. $y = \ln x + Ax^{-2} + B.$
K. $x = Ay^2 + B.$

7. Algae need nitrates to grow. In a lake, concentrations of nitrates in excess of 50 mg/L can cause algal blooms, where the algae grow uncontrollably. When such an uncontrolled growth of algae happens, the algae consume all of the oxygen in the water, and then all the fish die. Consider a lake with volume 500×10^9 L. Currently, the lake has a nitrate concentration of 10 mg/L. However, excess runoff from farms caused the nitrate concentration in a creek that feeds the lake to rise to 90 mg/L. The creek feeds the lake at a rate of 50×10^9 L/day, and the lake drains through a dam at the same rate. How long can this situation continue before the concentration of nitrates in the lake reaches 50 mg/L, putting the lake in danger of an algal bloom?

   A. Between one and two days.
   B. Between two and three days.
   C. Between three and four days.
   D. Between four and five days.
   E. Between five and six days.
   F. Between six and seven days.
   G. Between seven and eight days.
   H. Between eight and nine days.
   I. Between nine and ten days.
   J. Between ten and eleven days.
   K. Between eleven and twelve days.

8. Which of the following is an exact equation with general solution

$$y = x^2 + Cx^{-2}?$$

A. $(x^2 + 3y^2)\,dx + (6xy + 8y^2)\,dy = 0.$
B. $(2x + 1)\,dx - 2x^{-3}\,dy = 0.$
C. $(3xy^2 + 2x^2)\,dx + (3x^2y + 3y^3)\,dy = 0.$
D. $2y^2\,dx + (4xy + 6y^2)\,dy = 0.$
E. $(2x - y)\,dx + (8y - x)\,dy = 0.$
F. $y\,dx + (2x - 2x^{-3})\,dy = 0.$
G. $(4x - y)\,dx + 6y\,dy = 0.$
H. $2x\,dx + (3x + 2y)\,dy = 0.$
I. $(3x + 2y)\,dx + (2x + 3y)\,dy = 0.$
J. $(2xy - 4x^3)\,dx + x^2\,dy = 0.$
K. $(2xy^2 + 3x^2)\,dx + 2x^2y\,dy = 0.$

9. Find the general solution to the equation

$$\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 13\frac{d^2y}{dx^2} = 0.$$

A. $y = C_1e^{2x} + C_2xe^{2x} + C_3e^{2x}\cos(3x) + C_4e^{2x}\sin(3x)$.
B. $y = (C_1 + C_2x)e^{3x}\cos(2x) + (C_3 + C_4x)e^{3x}\sin(2x)$.
C. $y = C_1\cosh(2x) + C_2\sinh(2x) + C_3\cos(3x) + C_4\sin(3x)$.
D. $y = C_1\cosh(3x) + C_2\sinh(3x) + C_3\cos(2x) + C_4\sin(2x)$.
E. $y = C_1e^{3x} + C_2xe^{3x} + C_3e^{3x}\cos(2x) + C_4e^{3x}\sin(2x)$.
F. $y = (C_1 + C_2x)e^{2x}\cos(3x) + (C_3 + C_4x)e^{2x}\sin(3x)$.
G. $y = C_1e^{2x} + C_2xe^{2x} + C_3e^{3x} + C_4xe^{3x}$.
H. $y = C_1 + C_2x + e^{2x}(C_3\cos(3x) + C_4\sin(3x))$.
I. $y = C_1 + C_2x + e^{3x}(C_3\cos(2x) + C_4\sin(2x))$.
J. $y = C_1e^{3x} + C_2xe^{3x} + C_3\cos(2x) + C_4\sin(2x)$.
K. $y = C_1e^{2x} + C_2xe^{2x} + C_3\cos(3x) + C_4\sin(3x)$.

10. The function $y = e^{-x}$ is a solution to the differential equation

$$y'' - 7y' + 6y = 14e^{-x}.$$

Which of the following is also a solution to this differential equation?

A. $2\sinh x$.

B. $2\cosh x$.

C. $2e^{6x}$.

D. $2\cosh 6x$.

E. $2e^{-6x}$.

F. $2e^{-x}$.

G. $2\sinh 6x$.

H. $2\cos 6x$.

I. $2e^{x}$.

J. $2\sin 6x$.

K. $2e^{-7x}$.

11. Let

$$L = (D-1)(D-2)(2D-1) + D^3 - 40.$$

Compute

$$L\left(e^{3x}\right).$$

A. $-5e^{3x}$.
B. $-4e^{3x}$.
C. $-3e^{3x}$.
D. $-2e^{3x}$.
E. $-e^{3x}$.
F. 0.
G. e^{3x}.
H. $2e^{3x}$.
I. $3e^{3x}$.
J. $4e^{3x}$.
K. $5e^{3x}$.

12. One of the solutions of a linear homogeneous differential equation with constant coefficients is

$$y = 2e^{-2x}(2 + x^2 \cos(2x)).$$

What is the minimum possible order of the differential equation?

A. One.
B. Two.
C. Three.
D. Four.
E. Five.
F. Six.
G. Seven.
H. Eight.
I. Nine.
J. Ten.
K. Eleven.

13. Each of the choices presents three functions y_1, y_2, and y_3, which are solutions to a third-order homogeneous linear equation of the form

$$y^{(3)} + p_1(x)y'' + p_2(x)y' + p_3(x)y = 0,$$

where p_1, p_2, p_3 are continuous near $x = 0$. For exactly one of the choices, the solution to the initial value problem with the differential equation above and the initial conditions

$$y(0) = 2, \qquad y'(0) = 1, \qquad y''(0) = -2.$$

can be written in the form

$$y = C_1y_1 + C_2y_2 + C_3y_3$$

for constants C_1, C_2, and C_3. Which choice is it?

A. $y_1 = 2e^x$, $y_2 = e^{2x}$, $y_3 = e^{x+2}$.

B. $y_1 = \cos^2 x$, $y_2 = \cos 2x$, $y_3 = 2$.

C. $y_1 = 2e^{2x}$, $y_2 = 2e^{-2x}$, $y_3 = 0$.

D. $y_1 = \cosh 2x$, $y_2 = \sinh 2x$, $y_3 = -2e^{-x}$.

E. $y_1 = \cos^2 x$, $y_2 = \sin^2 x$, $y_3 = -2$.

F. $y_1 = 2e^{2x}$, $y_2 = -2e^{2x}$, $y_3 = 2e^{-2x}$.

14. Let y be the solution to the initial value problem

$$y^{(4)} + 2y'' + 2y = 0 \qquad \begin{aligned} y(1) &= 2, \\ y'(1) &= -3, \\ y''(1) &= -3, \\ y^{(3)}(1) &= 1. \end{aligned}$$

Compute $y^{(4)}(1)$.

A. -5.

B. -4.

C. -3.

D. -2.

E. -1.

F. 0.

G. 1.

H. 2.

I. 3.

J. 4.

K. 5.

15. Write your answer to this question on the free-response page.

Consider the general first-order differential equation

$$M dx + N dy = 0, \quad (\dagger)$$

where M and N are functions of x and y. In the homework, we determined how to find an integrating factor that depends only on x or an integrating factor that depends only on y. In this problem, we will find a condition under which (†) has an integrating factor which only depends on $2x - 3y$.

(a) (5 points) We're looking for a function $I(z)$ such that when we set $z = 2x-3y$ and multiply the above equation by $I(2x-3y)$, we get an exact equation. Multiplying by $I(2x - 3y)$, we get

$$M \cdot I(2x - 3y)\, dx + N \cdot I(2x - 3y)\, dy = 0.$$

Write down the criterion for exactness for this equation in terms of M and N, their partial derivatives, I, and I'. Be careful to apply the chain rule correctly when taking the partial derivatives of $I(2x - 3y)$.

(b) (5 points) Rearrange your equation from the previous part so that the terms involving M and N and their derivatives are on the left-hand side and the terms inolving I and I' are on the right-hand side. The right-hand side is a function that only depends on $2x - 3y$. Conclude that the left-hand side must also depend only on $2x - 3y$.

(c) (5 points) To simplify your work, let the left-hand side be $P(z)$, where $z = 2x-3y$. Solve a differential equation to find I in terms of P.

15. The text of this question is on the last page of the multiple-choice section.

Math 217 **Name:** ____________________
Midterm 2 **Student ID:** ____________________
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16. Make sure to show your work and justify your answers.

   (a) (10 points) Consider a cascade where water with a salt concentration of $4\,\mathrm{g/L}$ flows into Tank 1 at a rate of $5\,\mathrm{L/min}$. Tank 1 has $10\,\mathrm{L}$ of water with an initial salt concentration of $7\,\mathrm{g/L}$. The mixed water flows from Tank 1 into Tank 2 also at a rate of $5\,\mathrm{L/min}$. Under these conditions, the amount of salt in Tank 1 in grams at time t is $40 + 30e^{-t/2}$, where t is in minutes. Meanwhile, Tank 2 initially has $9\,\mathrm{L}$ of water with salt concentration $6\,\mathrm{g/L}$, and mixed water flows out of Tank 2 at a rate of $2\,\mathrm{L/min}$.

   Write down a differential equation with independent variable t and one dependent variable for the amount of salt in Tank 2.

   (b) (5 points) Write down a third-order homogeneous linear differential equation with constant coefficients with independent variable x and dependent variable y, such that as x approaches *negative* infinity, every solution approaches a constant and some solutions approach the constant solution $y \equiv 5$.