Differential Equations (Math 217) Midterm 1

September 20, 2016

- No calculators, notes, or other resources are allowed.
- There are 14 multiple-choice questions, worth 5 points each, and two hand-graded questions, worth 15 points each, for a total of 100 points.
- For the hand-graded questions, please turn in your solution to question 15 to one pile and your solution to question 16 to a separate pile.
- Write your name and student ID and circle your section on each page of your solutions to the hand-graded questions. The two questions span three pages, so you will do this three times.

Match the following differential equations to their solutions.

1. $x^2y'' + xy' - 4y = \ln(x^4)$.
2. $x^2y'' + 3xy' + y = 0$.
3. $xy' + 2y = xy^2$.
4. $xy' = 3 + 3x^4e^{3x-y}$.
5. $xy' = y + 3x^{1/3}y^{2/3}$.

A. $y = \frac{C_1 + C_2 \ln|x|}{x}$.

B. $y = C_1x^2 + \frac{C_2}{x^2} - \ln|x|$.

C. $y = \ln(x^3(C + e^{3x}))$.

D. $y = x(C + \ln|x|)^3$ or $y = 0$.

E. $y = \frac{1}{x + Cx^2}$ or $y = 0$.

What are the solutions to the following differential equations?

6. $y' + 2y = 3x^2 e^{-2x}$.

7. $xy' + 2y = 4x^2$.

8. $y' = 2xy - 2y$.

A. $y = \frac{1}{2}(1 + Ce^{2x-2})$.

B. $y = 3x^{1/2} + Cx^{-1/2}$.

C. $y = (2x + C)e^{x^2}$.

D. $y = (x^{4/3} + C)^2$.

E. $y = (2x^{3/2} + C)^2$.

F. $y = x^2 + \frac{C}{x^2}$.

G. $y = C(1 - x)^4$.

H. $y = Ce^{x^2}e^{-2x}$.

I. $y = Ce^{x+\frac{1}{2}x^2}$.

J. $y = x^2(C + \ln|x|)$.

K. $y = (x^3 + C)e^{-2x}$.

On November 12, 2014, the lander *Philae* detached from the spacecraft *Rosetta* and began free-falling towards the comet Churyumov–Gerasimenko from an initial altitude of 20 km (66 000 feet). According to Wikipedia, *Philae* has a mass of 100 kg, and the acceleration due to gravity on the surface of Churyumov–Gerasimenko is $0.001\,\mathrm{m/s^2}$.

9. Assume that *Philae* started its descent from rest and fell with constant acceleration. About how long did it take *Philae* to touch down on the comet? Note: $\sqrt{10} \approx 3$.

   A. 6 seconds.

   B. 30 seconds.

   C. 60 seconds.

   D. 300 seconds (5 minutes).

   E. 600 seconds (10 minutes).

   F. 3000 seconds (50 minutes).

   G. 6000 seconds (100 minutes).

   H. 30 000 seconds (8 hours).

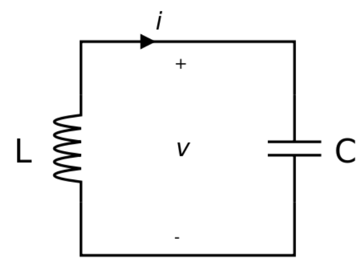
   I. 60 000 seconds (17 hours).

   J. 300 000 seconds (3.5 days).

   K. 600 000 seconds (7 days).

   Note: The correct answer to this question is off by a couple factors compared with what happened in reality. Both of our simplifying assumptions are not quite correct. *Philae* did not start its descent from rest, and the force of gravity is much smaller 20 km above the comet than it is on the surface, so the acceleration was not constant. But we did pretty well for not having a calculator.

For the next two questions, consider the following circuit diagram.



The squiggly line represents an inductor with inductance L, and the parallel lines represent a capacitor with capacitance C. From basic facts about electric circuits, we could figure out that the voltage V and current I across the capacitor satisfy the relationships

$$I = C\frac{dV}{dt}, \qquad V = -L\frac{dI}{dt}.$$

10. Use the second equation to substitute for V in the first equation, in order to obtain a second-order equation in terms of I and the constants C and L, but not in terms of V. Then rewrite it to look similar to the equation of a spring in class. Which of the following is a solution to this equation?

    A. $I = \cos(LCt)$.
    B. $I = \cos\left(\frac{C}{L}t\right)$.
    C. $I = \cos\left(\frac{L}{C}t\right)$.
    D. $I = \cos\left(\frac{1}{LC}t\right)$.
    E. $I = \cos\left(\sqrt{LC}t\right)$.
    F. $I = \cos\left(\sqrt{\frac{C}{L}}t\right)$.
    G. $I = \cos\left(\sqrt{\frac{L}{C}}t\right)$.
    H. $I = \cos\left(\sqrt{\frac{1}{LC}}t\right)$.

11. If you hook up the above circuit to an antenna, it will pick up radio waves at a frequency corresponding to your solution to the previous problem. However, you may want to listen to different radio stations, and your cellphone may want to adjust its frequency to communicate with the cell tower with less intereference, so you need to be able to tune the circuit. An inductor is a coil of wire, so it is difficult to tune the inductance L. However, a capacitor is two parallel metal plates, so you can tune the capacitance C by changing the distance between the two parallel plates. If you want to receive or transmit a signal at a frequency f, to what value should you tune your capacitance C?

    A. $C = 2\pi f L$.
    B. $C = \frac{2\pi f}{L}$.
    C. $C = \frac{L}{2\pi f}$.
    D. $C = \frac{1}{2\pi f L}$.
    E. $C = (2\pi f)^2 L$.
    F. $C = \frac{(2\pi f)^2}{L}$.
    G. $C = \frac{L}{(2\pi f)^2}$.
    H. $C = \frac{1}{(2\pi f)^2 L}$.

12. For exactly one of the following differential equations, there is more than one solution to the intial value problem with initial condition $y(1) = -1$. (Two of these solutions have different values everywhere, except at $x = 1$ itself.) Which differential equation is it?

    A. $y' = (x-1)^{4/3}(y-1)^{4/3}$.
    B. $y' = (x+1)^{-4/3}(y+1)^{4/3}$.
    C. $y' = (x+1)^{2/3}(y+1)^{2/3}$.
    D. $y' = (x-1)^{4/3}(y-1)^{-2/3}$.
    E. $y' = (x-1)^{2/3}(y+1)^{4/3}$.

13. You're hosting a barbecue with your friends on a 27 °C day. At 6pm, you go to grab a drink from the cooler and notice that all of the ice has melted and everything's warm. Also for some reason there's a thermometer in the cooler. As you close the cooler and run to get more ice, you notice that it reads 24 °C. When you come back with the ice, one of your friends asks you if you noticed what the temperature read.

"24 °C," you say, "Why?"

Your friend says, "Great, an hour ago, at 5pm, I noticed that all the ice had melted, so I put a thermometer in and measured the temperature to be 21 °C."

You ask, "Why didn't you tell me or get more ice?"

Your friend responds, "But then how would we figure out the coefficient of cooling?"

You make a mental note about whether to invite your friend to future barbecues. However, you're worried about how long the meat in the cooler has been warm. You'd ask your friend to figure it out, but you don't really want to talk to them anymore. As you know from physics, the temperature in the cooler remained at 0 °C until all the ice melted, at which point it started warming according to Newton's law of cooling. At what time did all the ice melt?

A. Between 7am and 8am.

B. Between 8am and 9am.

C. Between 9am and 10am.

D. Between 10am and 11am.

E. Between 11am and 12pm.

F. Between 12pm and 1pm.

G. Between 1pm and 2pm.

H. Between 2pm and 3pm.

I. Between 3pm and 4pm.

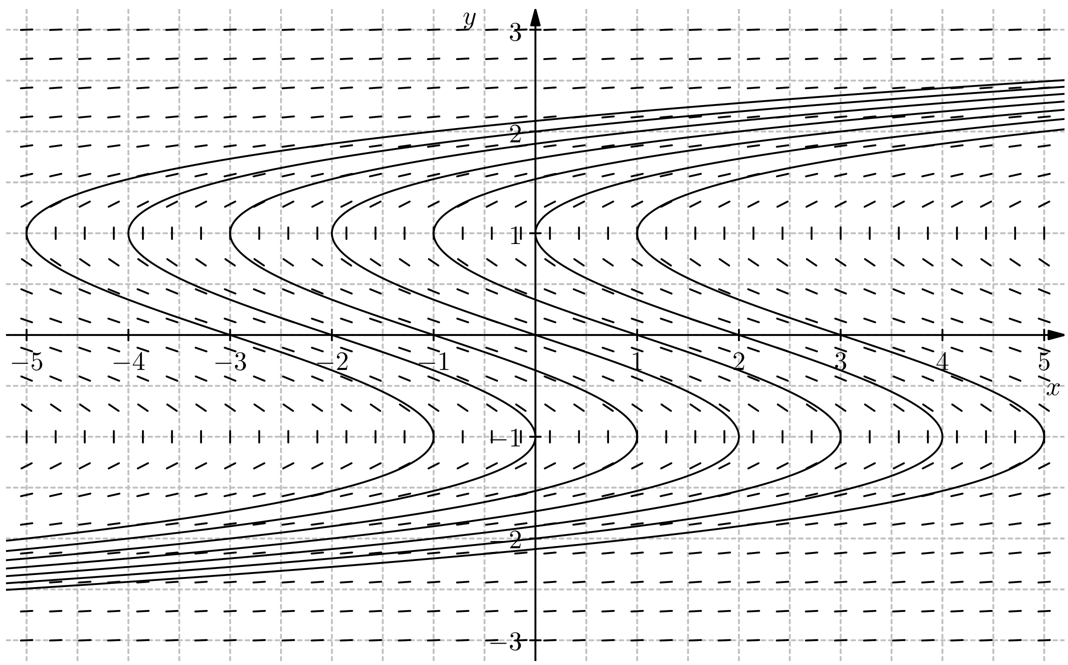
J. Between 4pm and 5pm.

K. Between 5pm and 6pm.

14. Consider the differential equation

$$(3y^2 - 3)y' = 1.$$

Some solution curves are shown below.



Find an implicit solution to this differential equation. Using your solution, the graphs, or likely both, find the largest domain on which the solution to the initial value problem

$$(3y^2 - 3)y' = 1, \qquad y(0) = -3.$$

is defined. The answer choices are on the next page.

Answer choices for question 14:

A. $(-\infty, 2]$.

B. $(-\infty, 4]$.

C. $(-\infty, 6]$.

D. $(-\infty, 8]$.

E. $(-\infty, 10]$.

F. $(-\infty, 12]$.

G. $(-\infty, 14]$.

H. $(-\infty, 16]$.

I. $(-\infty, 18]$.

J. $(-\infty, 20]$.

K. $(-\infty, \infty)$.

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15. (15 points) Find an explicit form for the the solution of the following initial value problem:

$$\frac{dy}{dx} = 4e^{2x-3y-1}, \qquad y(1) = 2.$$

Show your work.

16. In Webwork 2, you analyzed a modified logistic population model that took into account the minimum viable population of a species. In this problem, we will instead modify the logistic population model to take into account a carrying capacity that changes with time. Let P be the population of the species in thousands of individuals, and let t be time in months, with $t = 0$ representing the beginning of the year. Our model is

$$\frac{dP}{dt} = 0.05P\left(5 - 2\cos\left(2\pi \tfrac{t}{12}\right) - P\right).$$

(a) (1 point) Which expression in the model represents the carrying capacity?

(b) (1 point) The carrying capacity is periodic. What is its period? Include units.

(c) (1 point) Name a real-life factor that affects populations that could be modeled using such a periodic carrying capacity.

(d) (4 points) The slope field for this differential equation is on the next page. Sketch six solution curves, starting at $(0, 0), (0, 0.5), (0, 2), (0, 4), (0, 6)$, and $(0, 8)$.

(e) It looks like all of the solutions eventually approach the same periodic solution, which is the *steady state solution* and describes how the population will behave in the long run. Fill in the blanks.

i. (2 points) In the long run, the population will vary between approximately ____________ and approximately ____________ individuals.

ii. (2 points) Meanwhile, the carrying capacity varies between ____________ and ____________ individuals.

iii. (2 points) In the long run, the population will be at its lowest on around ____________ and at its highest on around ____________. (Enter the name of a month and a date, accurate to within a week or two.)

iv. (2 points) Meanwhile, the carrying capacity is at its lowest on ____________ and highest on ____________. (Enter the name of a month and date, accurate to within a day or two.)

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