Differential Equations Homework 9

Due October 31

Instructions

1. Write down the names of the people you worked with.
2. Write down any resources you used other than ones that most of your classmates would be familiar with, such as Wikipedia or Wolfram Alpha.
3. Write down at the top of your submission for part 1, separately, the number of hours it took you to complete this hand-graded assignment, and the number of hours it took you to complete the corresponding Webwork.
4. Write your name, Math 217, and the homework number.
5. Hand in your homework in class.
6. You'll be handing in your solutions to parts 1, 2, and 3 to separate piles to go to separate graders. Make sure they're on separate sheets of paper.
7. Unless directed otherwise, show enough work to convince a classmate that disagrees with you that you're right and they're wrong. Answers alone will usually receive no credit.

Problems

Part 1

1. Do problem 3.3.37 on page 171.
2. (a) Do problem 3.3.45 on page 171.
   (b) There is real-valued function that is a solution to the differential equation in 3.3.45. What is it?
3. Do problem 3.3.50 on page 171.

Part 2

4. (a) Do problem 3.3.48 on page 171. Hints: It will probably make things easier to keep the general solution in the complex form suggested in the book. Depending on the situation, either the polar or the rectangular form of α or β will make the computation easier than it would be otherwise. Also, even once you know what α and β are, it might still be a good idea to write α and β instead of the actual expressions until needed to make your formulas shorter.

   (b) Do the same problem except with initial conditions

$$y(0) = 0, \qquad y'(0) = 1, \qquad y''(0) = 0.$$

   Make sure to simplify your answer to a form similar to the answer to the book question, though your answer will also involve sine.

Part 3

5. In homework 7, you found a fourth-order homogenenous linear equation with constant coefficients so that $y = x\cos 2x$ is one of its solutions, by computing the derivatives of $x\cos 2x$ and cancelling terms.

   (a) For this problem, we'll use what we've learned to find such an equation an easier way. Determine the roots of the characteristic polynomial, expand it out, and write down the differential equation.

   (b) Write down the general solution to your differential equation.

6. In homework 7, you found a first-order homogenenous linear equation so that $y = x\cos 2x$ is one of its solutions. However, in the standard form,

$$y' + p(x)y = 0.$$

   the coefficient $p(x)$ was not continuous at $x = 0$. We'll see why that has to be the case.

   (a) There is a function that is a solution to any first-order homogeneous linear equation of the form

$$y' + p(x)y = 0.$$

   What is this function, and what initial condition does it satisfy at $x = 0$?

   (b) Let's say that $y = x\cos 2x$ satisfies a differential equation of the form

$$y' + p(x)y = 0.$$

   What initial condition does $y = x\cos 2x$ satisfy at $x = 0$?

   (c) Why does the existence and uniqueness theorem (page 151) imply that $p(x)$ can't be continuous at $x = 0$?

(d) Find a second-order homogeneous linear equation in standard form

$$y'' + p_1(x)y' + p_2(x)y = 0.$$

so that $y = x\cos 2x$ is one of its solutions and so that the coefficients $p_1(x)$ and $p_2(x)$ are continuous at $x = 0$. Hint: I suggest doing it with $p_2(x) \equiv 0$, using the same technique as in Homework 7. Don't forget to check that your coefficient is continuous at $x = 0$.

(e) Find the general solution to your equation from the previous part on the interval near $x = 0$ where the coefficients are continuous.

(f) If you followed the hint in the previous part, then you found an equation where $p_1(x)$ and $p_2(x)$ are continuous at $x = 0$, but $p_1(x)$ is not continuous everywhere. For this part, find a second-order homogeneous linear equation in standard form so that the coefficients p_1 and p_2 are continuous everywhere. Essential hint:

$$y'' = y''\frac{(y')^2 + y^2}{(y')^2 + y^2} = \frac{y''y'}{(y')^2 + y^2}y' + \frac{y''y}{(y')^2 + y^2}y.$$

For this problem, you are not required to simplify the equation, but you do need to justify why the denominator is never zero.

In terms of practical applications, often the coefficients of a differential equation are parameters that we can tune so that the system behaves the way we want. In some applications, we can tune the coefficients in real time, that is, we can make them be functions of x rather than just constants. However, generally, we can't make the parameters infinite.

Part don't turn this one in yet

7. To keep the length of the problem set down, the following problem about $y = x\sin 2x$ will be part of homework 10. **Don't turn it in this week.** However, it follows the line of reasoning of the previous problems, so you might want to get started on it now.

(a) Write down a homogeneous linear equation with constant coefficients so that $y = x\sin 2x$ is one of its solutions.

(b) i. Apply the reasoning of the problem 6 to answer whether the existence and uniqueness theorem allows $y = x\sin 2x$ to be a solution to a first-order homogeneous linear differential equation

$$y' + p_1(x)y = 0,$$

where $p_1(x)$ is continuous at $x = 0$.

ii. Does the existence and uniqueness theorem allow $y = x\sin 2x$ to be asolution to a second-order homogeneous linear differential equation

$$y'' + p_1(x)y' + p_2(x)y = 0,$$

where the coefficients $p_1(x)$ and $p_2(x)$ are continuous at $x = 0$?

iii. What stops us from using the trick from problem 6(f) to construct such an equation?

iv. Does the existence and uniqueness theorem allow $y = x\sin 2x$ to be a solution to a third-order homogeneous linear equation?

v. What about a fourth-order one?

(c) For the smallest order n where you determined it is possible to do so, write down an nth order homogeneous linear differential equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0,$$

where all of the coefficients are continuous at $x = 0$, so that $y = x \sin 2x$ is one of the solutions.