Differential Equations Homework 5

Due October 3

Instructions

1. Write down the names of the people you worked with.
2. Write down any resources you used other than ones that most of your classmates would be familiar with, such as Wikipedia or Wolfram Alpha.
3. Write down at the top of your submission for part 1, separately, the number of hours it took you to complete this hand-graded assignment, and the number of hours it took you to complete the corresponding Webwork.
4. Write your name, Math 217, and the homework number.
5. Hand in your homework in class.
6. You'll be handing in your solutions to parts 1, 2, and 3 to separate piles to go to separate graders. Make sure they're on separate sheets of paper.
7. Unless directed otherwise, show enough work to convince a classmate that disagrees with you that you're right and they're wrong. Answers alone will usually receive no credit.

Problems

Part 1

1. (a) Solve

$$\frac{dy}{dx} + 2xy = 2kx,$$

where k is a constant, using an integrating factor. Your solution will be in terms of k and an integration constant C.

(b) This is equation is also separable. Solve it using separation of variables. Make sure your answer matches the previous part.

(c) Consider the equation

$$\frac{dy}{dx} + P(x)y = kP(x),$$

where k is a constant and $P(x)$ is an arbitrary function of x. Notice that the equation in part a is a special case of this equation, and use that to guide you. But first, without solving the differential equation, show that the constant function $y = k$ is a solution.

(d) Solve this equation using an integrating factor. Your answer should be in terms of k, P, and an integration constant C. ($\int P(x)\,dx$ counts as an expression in terms of P.)

(e) Now solve this equation using separation of variables. Make sure your answer matches with the previous part.

(f) Check that when you plug in $P(x) = 2x$ into your answer from the previous part, you get your answer from part a.

Part 2

2. Do problem 1.5.40 on page 54 of the textbook. If needed, use problems 38 and 39 as warmups. For part (b), if you are unfamiliar with induction and want to keep it that way, you may instead compute x_1, x_2, x_3, x_4, and x_5 and verify that your expressions match the one given in the book, and then go back to working with general x_n for the rest of the problem.

3. For a more realistic situation modeled by the previous problem, consider a stream a long time after it has rained. Such a stream has pools of water blocked by rocks, branches, or other debris, with a small trickle from one pool to the next. If someone spills some contaminant into one of the pools, what happens downstream?

   (a) We first need to figure out how problem 2 works more generally. Assume that the tanks each contain a constant volume V of liquid, that the flow rate between consecutive tanks is a constant r, and that the initial amount of contaminant in Tank 0 is x_{init}. Write down a formula for x_n. Note: Keep in mind that in problem 2, although Tank 0 contains 1 gal of water, the important part is that it contains 2 gal of liquid.

   Hints: If all else fails, you can get the answer by redoing problem 2, replacing 2 gal with V, 1 gal/min with r, and 1 gal with x_{init}. Alternatively, track how these values would affect your solution throughout problem 2. Another tool you could use is to compare and check units. Yet another option is to think about scaling. If you double x_{init} but keep everything else the same, what would you expect to happen to x_n? If you double the flow rate r but keep everything else the same, what would you expect to happen to the time it takes for x_n to reach a particular value? What happens to the volume V, flow rate r, and contaminant amounts x_{init} and x_n if we put two identical cascades A and B of tanks next to each other, and pretend that Tank nA and Tank nB are actually a single Megatank n of the new, larger, cascade? As you work, unitless expressions are often a sign you're on the right track, and here, tr/V is a unitless expression, so hopefully you'll see it pop up, and maybe using it will simplify some of your expressions.

   (b) At what time t_n does the amount of contaminant x_n of Tank n reach its maximum? What is its maximum value M_n? Use Stirling's approximation, and confirm that your units match and that when you plug in the numbers from problem 2, your answers agree. Feel free to solve this problem either directly from the previous part or by continuing tracking how V, r, and x_{init} would affect your solution to problem 2.

   (c) Getting back to the stream, let's say it's a fairly small stream. Each of the pools has a volume of about 30 000 L, and the trickle rate between them is about 5000 L/hr. Someone (wasn't me) accidentally spilled a gallon of bleach into one of the pools, henceforth called Pool 0. The volume of the bleach is insignificant, so the relevant fact for this problem is that a gallon of bleach has about 200 g of sodium hypochlorite. A sodium hypochlorite concentration of 0.6 mg/L or more is bad news for fish. How long after the bleach was spilled will bad news for fish be over in Pool 0? Include units in your answer.

(d) Numbering the pools consecutively from Pool 0, which pools will have bad news for fish at some point?

(e) Approximately how long after the bleach was spilled will bad news for fish be over throughout the entire stream? To approximate, assume that in the last pool to have bad news for fish, the sodium hypochlorite concentration just barely exceeded 0.6 mg/L, so the bad news for fish was (comparitively) brief and happened right around when the sodium hypochlorite concentration reached its maximum value in that pool. Include units in your answer.

(f) Confirm your answer to the previous part by using your formula for x_n and Wolfram Alpha or some other tool to solve for t exactly. In the last pool to have bad news for fish, when did it start and end, and how long did it last? Include units in your answer.

(g) (Optional, useful for checking your work.) The pool where bad news for fish lasted the longest was Pool 7. Check using Wolfram Alpha that, in Pool 7, bad news for fish began at around $t \approx 28$ hours and ended at around $t \approx 60$ hours, lasting for a total of about 32 hours.

Part 3

4. Do problem 1.6.57 on page 70 of the textbook. Since $\ln y$ appears in the original problem, you can assume that $y > 0$. Depending on your tastes, you might prefer to do the next problem first as a warmup, and then do this one.

5. Do problem 1.6.58 on page 70 of the textbook. Do check your answer in the back of the book, but make sure your work demonstrates your understanding.