Differential Equations Homework 4

Due September 23

Instructions

1. Write down the names of the people you worked with.
2. Write down any resources you used other than ones that most of your classmates would be familiar with, such as Wikipedia or Wolfram Alpha.
3. Write down at the top of your submission for part 1, separately, the number of hours it took you to complete this hand-graded assignment, and the number of hours it took you to complete the corresponding Webwork.
4. Write your name, Math 217, and the homework number.
5. Hand in your homework in class.
6. **You'll be handing in your solutions to parts 1, 2, and 3 to separate piles to go to separate graders. Make sure they're on separate sheets of paper.**
7. Unless directed otherwise, show enough work to convince a classmate that disagrees with you that you're right and they're wrong. Answers alone will usually receive no credit.

Problems

Part 1

1. You're in charge of purchasing fiberoptic cable for a rural long-haul aerial line.
   (a) Do exercise 1.4.69 on page 43 of the book concerning the catenary.
   (b) The textbook problem gives you the tension $T = T(0)$ at $x = 0$, but, at a general point along the cable,
   $$T(x) = T(0)\sqrt{1+v^2}.$$
   Write down an equation relating the maximum tension of the cable T_{max} to the weight density of the cable ρ, the halfway distance between support towers L, and the catenary shape parameter a.
   (c) Each time you connect two fiberoptic cables together, some signal is lost at the boundary, so you definitely want to buy the right amount of cable the first time. Compute the length l of the cable in the textbook problem, in terms of a and L.

(d) When the tension is high and the cable is light, you'd expect the cable to be almost straight. In this situation, is a very large or very small? Check your answer to the previous part by verifying that, in this situation, the length of the cable is just a tiny bit longer than the distance between the supports. Note that, from the Taylor series, when z is small and positive, $\sinh z \approx z$, and $\sinh z$ is slightly larger.

(e) You consult a catalog to determine the technical specifications of the cables that are available to you. For the purposes of this problem, your catalog is `https://en.wikipedia.org/wiki/All-dielectric_self-supporting_cable`, which tells you that the lightest ADSS cables are 220 kg/km, and the strongest can support a tension of 25 kN. Your company can afford to buy the lightest and strongest cables. However, the region you're putting these cables has a lot of wind and an occasional ice storm. To have a margin of error in case your cable gets loaded with a lot of heavy ice, company policy is to have the maximum tension in a cable hanging in ideal conditions (with no wind or ice) to be at most 5% of the cable's limit.

Write down your values for ρ and T_{max}. Include units. Be careful: Your catalog gives units of mass/distance, but, in the book, ρ has units weight/distance, so make sure to convert between mass and weight appropriately.

(f) If the distance between the support towers is 200 m, how much cable do you need to cover 1 km? What if the distance between the support towers is 500 m? 1 km? Include units. Hints: There isn't a closed form solution to your equation from part b, so you'll need to use some software, such as Wolfram Alpha, in order to solve for a.

(g) If you did the previous part correctly, you eventually ran into a problem. What was the mathematical issue? What would be the issue in real life? Why can't we solve the real-life issue just by lengthening the cable in order to reduce the tension?

Part 2

2. In this class, we've moved up from solving equations whose solutions are numbers to solving equations whose solutions are functions. These differential equations model real-life systems, and the solutions model how those systems behave. However, eventually, we want not only to passively observe a system, but to control it, like the triple pendulum on a cart in the video from a few weeks ago.

   We model such a situation with a differential equation that has some terms that model how the system behaves naturally, and some terms that model our ability to control the system. We get to choose what those control terms are, and we make our choice so that the solutions to the differential equation behave the way we want. In doing so, we move up from solving for functions to solving for the differential equations themselves.

   In this problem, we'll explore some simplified examples.

   (a) Consider a ship. Let m be its mass, and let $v(t)$ be its velocity. There are two forces acting on the ship. The first is viscous friction, which is proportional to the velocity of the ship, so we can write the magnitude of this force as kv for some constant k. As one expects, friction slows the ship down. The second force comes from the motor. We get to control what this force is. Call it F_m. Write down a differential equation for v that models this situation, in terms of m, k, and F_m.

(b) The ship has a cruise control system, which you get to design. The skipper selects a desired cruise speed v_c. Design an expression for F_m, in terms of any of the above quantities, so that the ship's speed approaches v_c in the long run. Write down the resulting differential equation. Hint: If the ship's speed is already v_c when the skipper turns on the cruise control system, what would you like the ship to do?

(c) Find the general solution to your differential equation, and verify that all solutions approach v_c in the long run.

(d) Now consider a spaceship in deep space. Again, let m be its mass, and let $v(t)$ be its velocity. In this case, there is no friction, so the only force comes from your thrusters, which you control. Let F_t be this force. Write down a differential equation for v that models this situation, in terms of m and F_t.

(e) Again, design a cruise control system for the spaceship so that, in the long run, it approaches a set speed v_c. Write down an expression for F_t in terms of any of the above quantities, and then write down the resulting differential equation. Verify that all solutions approach v_c.

Note: These examples give the right idea, but they are a bit unrealistic. A cruise control system for a boat does not know if there's a fishing net being dragged behind the boat, increasing the friction. As a result, when designing the cruise control system, in real life we do not have access to the parameter k. Likewise, the passengers are not weighed before getting on the boat, so the cruise control system does know what m is either.

However, the control system does have access to v, and real-life control systems can use use more advanced techniques to decide how much force the motor provides based on v alone. For example, the jack-of-all-trades PID controller, which many of you will see in your engineering classes, can base the force F_m off of things such as the integral of the error, $v - v_c$, over the past ten seconds. Analyzing the behavior of the boat with such a control system is beyond the tools we have in this class, but it is the way real-life control systems, such as the cruise control system in a car, work without knowing the weights of the passengers or whether you're going uphill or downhill.

Part 3

3. An *estuary* is a body of water where fresh water from a river mixes with salt water from the ocean, as a result of which the salinity of the estuary is somewhere in between. As a result, estuaries are a fairly unique ecological niche. We'll make and analyze a fairly simplistic model of an estuary.

   A freshwater river flows into an estuary with a constant flow rate of 400×10^6 L/hr. During high tide, additionally, seawater from the ocean, with salt concentration 35 g/L, flows into the estuary at a rate of 200×10^6 L/hr. At low tide, the mixed water in the estuary flows into the ocean at a rate of 1000×10^6 L/hr. As a result, the volume of the estuary changes with time. The average volume of our small estuary is $20\,000 \times 10^6$ L. Low tide and high tide each last for about 6 hours and 13 minutes, but for this problem, round it off to 6 hours.

   (a) Let $V(t)$ be the volume of the estuary, with t representing time since the beginning of low tide. Compute $V(t)$. Either include units in your expression or write down the units of all the variables.

(b) Let $x(t)$ be the amount of salt in the estuary. Write down a differential equation for how x behaves during low tide. Again, either include units in your expression, or write down the units of variables you didn't already talk about.

(c) Solve your differential equation using an integrating factor. (Units optional.)

(d) Let x_h be the amount of salt at the end of high tide, that is, at the beginning of low tide. Use your solution to compute the amount of salt at the end of low tide, after some of the salt has flowed into the ocean, in terms of x_h.

(e) Now, write down a differential equation for how x behaves during high tide. Include units in your equation.

(f) Solve your differential equation. (Units optional.)

(g) Let x_l be the amount of salt at the end of low tide, that is, at the beginning of high tide. Using your solution or by any other means, compute the amount of salt at the end of high tide, after salt water from the ocean has entered the estuary, in terms of x_l. Either include units, or, if you wrote down units for your variables earlier, make sure your expression is consistent with the units you wrote down for x.

(h) If there hasn't recently been any unusually high rainfall or a drought that would have affected the flow rate of the river, we'd expect the estuary to be at a steady state, where the amount of salt at the end of high tide is always the same value x_h, and the amount of salt at the end of low tide is always the same value x_l. Find x_h and x_l, and include units in your final answer.

(i) Convert your two answers in the previous part to *concentrations* of salt. Include units, and compare the two concentrations with each other and with the salt concentration of the ocean.