Differential Equations Homework 3

Due September 16

Instructions

1. Write down the names of the people you worked with.
2. Write down any resources you used other than ones that most of your classmates would be familiar with, such as Wikipedia or Wolfram Alpha.
3. Write down at the top of your submission for part 1, separately, the number of hours it took you to complete this hand-graded assignment, and the number of hours it took you to complete the corresponding Webwork.
4. Write your name, Math 217, and the homework number.
5. Hand in your homework in class.
6. **New. You'll be handing in your solutions to parts 1 and 2 to separate piles to go to separate graders. Make sure they're on separate sheets of paper.**
7. Unless directed otherwise, show enough work to convince a classmate that disagrees with you that you're right and they're wrong. Answers alone will usually receive no credit.

Problems

Part 1

1. (20 points) In homework 1, we discussed functions whose derivatives are themselves. In this problem, we're going to look at functions $f(x)$ whose derivatives are their multiplicative inverses, that is, $f'(x) = \frac{1}{f(x)}$.
   (a) (2 points) Find a function satisfying this property.
   (b) (3 points) Find all functions satisfying this property.
   (c) (3 points) Sketch the graphs a few of these functions.
   (d) (2 points) Find the function satisfying this property such that $f(1) = -2$.
   (e) (1 point) Include the graph of this function on your sketch, and label the point $(1, -2)$.
   (f) (2 points) What is the largest domain on which the function from part d is defined?

(g) (2 points) Find another function with the same domain as your answer to part d that satisfies the property $f'(x) = \frac{1}{f(x)}$. Write down this function, and include its graph on your sketch.

(h) (3 points) Together, the graphs of your answers to parts d and g form a single curve. What is its equation, and what is the name of a curve of this shape?

(i) (2 points) Why isn't this entire curve a valid answer to part a?

2. (20 points) Consider the family of parabolas $y = Cx^2$.

   (a) (3 points) Sketch this family of curves. Include at least seven curves, and make sure to include curves for positive, zero, and negative values of C.

   (b) (4 points) Write down a differential equation that all of these curves satisfy. (Your differential equation should not involve C.)

   Given a family of curves, in applications it's often useful to consider a second family of curves that are everywhere perpendicular to the first family. For example, topographical maps have *countour lines*, which show where the land is at a particular elevation. For example, the 200 ft contour line shows where you'd go if you started at 200 ft and walked without ever going uphill or downhill. Streams, on the other hand, flow directly downhill, so stream paths are always perpendicular to the contour lines. Likewise, electric field lines are perpendicular to the curves of constant electric potential. You could imagine the family of parabolas above to be the path that water flows from the top of a hill located at the origin, for example.

   (c) (3 points) Add to your sketch a second family of curves that is everywhere perpendicular to the family of parabolas you've drawn.

   (d) (4 points) Use the differential equation you wrote for the parabolas to write down a differential equation for this new family of curves.

   (e) (3 points) Find an implicit solution of this differential equation.

   (f) (3 points) What geometric shape are the curves in the second family?

3. (20 points) Consider the differential equation

$$y' = x^2(y+1)^{4/5}.$$

   (a) (5 points) Find all solutions to this differential equation. Don't forget the singular solution that the method of separation of variables misses because of division by zero.

   (b) (5 points) Find two different solutions to this differential equation that satisfy $y(0) = -1$.

   (c) (5 points) The existence and uniqueness theorem guarantees that solutions to the initial value problem $y(a) = b$ exist and are unique, at least near $x = a$, as long as ______________. Fill in the blank with a condition in terms of a and/or b. (As always, show your work.) Comment on what this means for part b.

   (d) (5 points) Find two different solutions to this differential equation that satisfy $y(0) = 0$.

   (e) (5 points) Reconcile your answers to parts c and d, and find the largest domain on which the solution to the initial value problem

$$y' = x^2(y+1)^{4/5}, \qquad y(0) = 0.$$

   exists and is unique.

   Hint: My guess is that you'll find playing around with this problem in Geogebra helpful.

Part 2

4. (20 points) In this problem, we consider the Newton's law of cooling equation for the temperature T of an object in an environment at ambient temperature A.

$$\frac{dT}{dt} = -k(T - A). \tag{1}$$

   (a) (3 points) Solve this differential equation using our technique for solving separable differential equations. Make sure your general solution includes the constant solution.

   (b) (5 points) We will now solve this differential equation a different way. Let y be the difference between the object's temperature and the environment's temperature.

      i. (1 point) Write down an equation relating T and y.

      ii. (1 point) Write down an equation relating $\frac{dT}{dt}$ and $\frac{dy}{dt}$.

      iii. (1 point) Use substitution to rewrite equation (1) in terms of y. Your differential equation should involve y and k, but not T.

      iv. (1 point) In class, we saw the solution to the exponential growth/decay equation. Use that to directly write down a general solution y for your differential equation.

      v. (1 point) Substitute back to write your solution in terms of T, k and t, but not y.

The safe cooking temperature for chicken is 75 °C. Using a meat thermometer, you took some chicken thighs out of the oven when they were at exactly 75 °C. You happened to leave the meat thermometer in the chicken, so you know that, three minutes later, the chicken cooled to 51 °C. Meanwhile, your kitchen was pretty warm, at 27 °C, because of all the baking.

   (c) (3 points) Complete the following chart.

Time (min)	$T - A$	T
0		75 °C
3		51 °C
6		
9		
12		
15		

   (d) (3 points) According to the internet, 45 °C is a safe temperature for food not to burn your mouth. Based on your chart, in what three-minute interval after the chicken is taken out of the oven will the food become safe to eat?

   (e) (3 points) Using a calculator, at what exact time will the chicken reach 45 °C?

   (f) (3 points) If you did the previous parts correctly, 45 °C should be exactly halfway between the temperature at 3 minutes and the temperature at 6 minutes. However, the time you found in part e is not exactly halfway between 3 minutes and 6 minutes. Is it earlier or later than the halfway point? Why would you expect that to be the case, even if you did not have a calculator to compute the exact value?

5. (20 points) In this question, we'll expand a bit on the method we used in problem 4(b).

(a) (10 points) Consider a differential equation

$$\frac{dx}{dt} = f(t, x).$$

Let's say that the function $f(t, x)$ is one for which we know how to solve this equation and can find solutions $x = s(t)$. Using problem 4(b) to guide you, find a solution to the differential equation

$$\frac{dx}{dt} = f(t, x - a),$$

where a is a constant. Your work should be in terms of f, and your answer will be in terms of s.

Hint: Let $y = x - a$.

(b) (10 points) We'll take this idea one step further. Consider the differential equation

$$\frac{dx}{dt} - kx = -kt + 1,$$

where k is a constant.

i. (2 points) Is this equation separable? Why or why not?

ii. (2 points) Rearranging, this equation is

$$\frac{dx}{dt} = k(x - t) + 1.$$

This form of the equation suggests letting $y = x - t$. Find an equation relating $\frac{dy}{dt}$ and $\frac{dx}{dt}$.

iii. (2 points) Rewrite the differential equation in terms of y, t, and k, but not x.

iv. (2 points) This new equation should be separable, and, in fact, you should know its solution. Write down the general solution y to your equation, in terms of k and t.

v. (2 points) Write down the general solution x to the original equation, in terms of k and t.

If needed, pick a value of k and use Geogebra to check your answer.