DIFFERENTIAL EQUATIONS HOMEWORK 2, DUE SEPTEMBER 9

Instructions

(1) Write down the names of the people you worked with.
(2) Write down any resources you used other than ones that most of your classmates would be familiar with, such as Wikipedia or Wolfram Alpha.
(3) **Changed.** Write down, separately, the number of hours it took you to complete this hand-graded assignment, and the number of hours it took you to complete the corresponding Webwork. (If you could go back and separate the hand-graded hours from the Webwork hours for homework 1, that'd be helpful as well.)
(4) Write your name, Math 217, and the homework number.
(5) Hand in your homework in class.
(6) **New.** Even though homeworks 1 and 2 are due on the same day, you'll be handing them in to separate piles to go to separate graders. If parts of different homeworks are on the same sheet of paper, you'll run into trouble.
(7) **New.** Unless directed otherwise, show enough work to convince a classmate that disagrees with you that you're right and they're wrong. Answers alone will receive no credit.

Problems

(1) Consider the following equations:
  A. $y' = 3x^2(y^2 + 1)$.
  B. $y(0) = 1$.
  C.
$$\begin{cases} y' = 3x^2(y^2+1) \\ y(0) = 1. \end{cases}$$
  D. $y = \tan(x^3 + C)$.
  E. $C = \frac{\pi}{4}$.
  F. $y = \tan(x^3 + \frac{\pi}{4})$.
  G. $y = \tan(x^3) + \frac{\pi}{4}$.
  H. $y = \tan(x^3)$.
  (a) Without showing your work, write down which of the above are
    (i) a differential equation,
    (ii) an initial value problem,
    (iii) an initial condition,
    (iv) a solution to the differential equation,
    (v) a general solution to the differential equation,
    (vi) a particular solution to the differential equation,
    (vii) a solution to the initial value problem.
  Note: Some parts will have multiple answers, in which case you should write down all of them. Some answers might be used more than once, and some might not be used at all.
  (b) Justify your answer to problem 1(a)iv. Doing so requires
    (i) showing that every answer you wrote down for 1(a)iv is correct, and

(ii) showing that every answer you did not write down for 1(a)iv is incorrect.

(c) What is the domain of the solution to the intial value problem above?

(d) What is the solution to the initial value problem if we replace $y(0) = 1$ with $y(0) = 0$?

(e) What is the solution to the initial value problem if we replace $y(0) = 1$ with $y(0) = -1$?

(f) Qualitatively, what happened to the domain of the solution as we changed the intial value problem from the original to the one in 1d, and then to the one in 1e?

Note: This equation appears in problem 1.1.25 in the book. It, and the graph in its answer in the back of the book, could prove useful for checking your work. However, for most people, looking at it before you're ready to check your work is not the best idea in terms of doing well in the rest of this course.

(2) According to Wikipedia, propofol is one of the drugs commonly used for general anesthesia. Also according to Wikipedia, its biological half-life is 30 to 60 minutes. For this problem, let's split the difference and assume that we have an average patient for whom the half-life is 45 minutes. In real life, biological half-life is somewhat complicated, because, in addition to being eventually processed by the liver, the drug moves from the blood stream to other tissues where it's no longer effective, and an accurate model would have to take all of these effects, each of which have different rates, into account. This task would be reasonable for the lucrative job of a pharmacokineticist, but not for a homework problem. So, we will make a reasonable simplifying approximation that, if given a dose of propofol, the amount of propofol in the body P will decay exponentially according to the equation

$$\frac{dP}{dt} = -kP$$

that we've seen in class.

(a) Based on the above assumptions, compute k. Include units.

(b) In real uses, propofol is not administered all at once, but rather gradually through an IV. According to a Google search, a usual infusion rate is 100–200 mcg/kg/minute. In other words, for every kilogram that the patient weighs, 100–200 micrograms of propofol is administered intraveneously every minute. Assume our patient weighs 80 kilograms, and let r be the rate at which the drug is administered, with units micrograms/minute. Compute the range of values for r based on the above assumptions. Include units.

(c) Amend the differential equation above to also model the IV injection by including r. Your equation should be in terms of P, k, and r.

(d) Rearrange your equation to be of the form

$$\frac{dP}{dt} = -k(P - S).$$

In other words, your equation should be of the above form, where S is some expression in terms of k and r.

(e) Notice that this equation is exactly the same as Newton's law of cooling, but with different variables. Based on your understanding of cooling, after a while, the amount of propofol in the patient's body should approach what steady state value? Answer in terms of k and r.

(f) With an infusion rate of 100 mcg/kg/minute, what will be the steady state amount of drug in the patient's body? What if you increase the infusion rate to 200 mcg/kg/minute? Include units.

(g) If you wanted the amount of propofol in the patient's body to become 1 gram, at what rate r should you adminster the drug? Include units.

(3) This problem is about the differential equation

$$\frac{dx}{dt} = x - \tfrac{1}{2}t - 8. \tag{1}$$

(a) Produce a slope field plot for this equation. Follow the following general guidelines.
 (i) You may produce your diagram by hand, using a computer, or some combination thereof.
 (ii) You may only use software that is freely available to all students in the class. The Geogebra slope field plotter is great. A slope field plotter you're convinced is better that's available on an obscure website is also great, provided you mention the resource as per the homework instructions. Proprietary software that is freely available to WashU students is also great. Personal software that you won in some math contest is not so great.
 (iii) Label and mark the axes, and label all important features.
 (iv) Include at least five solution curves.
 (v) Make sure the graph is big enough so that it's not too cluttered.

 In addition, follow these specific guidelines for this problem:
 (i) Make sure your slope field includes several places where the slope is horizontal.
 (ii) Have your slope field be centered around the x-axis, but not necessarily the y-axis.
 (iii) It looks like as t becomes very negative, the solutions approach a line, which is itself a solution. Make sure your solution curves include this line and some trajectories above and below it.

(b) Find the equation of the line from part 3(a)iii.

(c) Verify that this line is a solution of the differential equation (1).

(d) Come up with an initial condition so that this line is a solution of the initial value problem with the differential equation (1) and your initial condition.

(e) Come up with a different initial condition so that this line is a solution of the initial value problem with the differential equation (1) and your initial condition.

(4) This problem is about the differential equation

$$y' = x^2 - y - 2. \tag{2}$$

(a) Produce a slope field plot for this equation. Follow the same general guidelines as in problem 3a. As for the specific guidelines,
 (i) Make sure your slope field includes several places where the slope is horizontal.
 (ii) Have your slope field be centered around the origin.
 (iii) It looks like as x becomes very positive, the solutions approach a parabola, which is itself a solution. Make sure your solution curves include this parabola and some trajectories above and below it.

(b) Play around in Geogebra until you have a good sense of what the graph of this parabola looks like. Use that information to find the equation of this parabola. In the input line in Geogebra, you can write an equation like $y = -x^2$ and it will graph it for you. You can use this feature to help you see whether or not you have the right equation.

(c) Verify that this parabola is a solution of the differential equation (2).

Note: This equation appears as problem 1.3.9 in the book, and the slope field is drawn there. In addition, in the answers in the back of the book, several more trajectories are drawn. I encourage you to use these to check your work, but you may not copy the pictures in the book (including tracing by hand or other clever means) for problem 4a.